INSTRUMENTAL WEIGHTED VARIABLES - $\sqrt{n}$-CONSISTENCY

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PRAGUE

Content of contribution

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Task is to find for the data

$$\begin{bmatrix} Y_1, & X_{11}, & X_{12}, & \cdots, & X_{1p} \\ Y_2, & X_{21}, & X_{22}, & \cdots, & X_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ Y_n, & X_{n1}, & X_{n2}, & \cdots, & X_{np} \end{bmatrix}$$

a relation between

the response variable $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ \vdots \\ Y_n \end{bmatrix}$ (on one-hand-side)

and

explanatory variables $X = \begin{bmatrix} X_{11}, & X_{12}, & \cdots, & X_{1p} \\ X_{21}, & X_{22}, & \cdots, & X_{2p} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ X_{n1}, & X_{n2}, & \cdots, & X_{np} \end{bmatrix}$ (on the other).

Let's consider

REGRESSION MODEL

$$Y_i = X_i'\beta^0 + \varepsilon_i = \sum_{j=1}^{p} X_{ij}{\beta^0}_j + \varepsilon_i, \quad i = 1, 2, ..., n$$

or in the matrix form

$$Y = X\beta^0 + \varepsilon$$

where

$$\beta^0 = \begin{bmatrix} \beta_1^0 \\ \beta_2^0 \\ \vdots \\ \beta_p^0 \end{bmatrix}$$ are regression coefficients and $$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \vdots \\ \varepsilon_n \end{bmatrix}$$ disturbances .

The goal is to estimate unknown regression coefficients.

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Content of contribution

1. Basic framework and goal
2. The most frequently used methods - in the past and today
3. Underestimated deficiency of classical methods
4. Robustification of the Instrumental Variables

The ORDINARY LEAST SQUARES

$$\hat{\beta}^{(OLS,n)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} (Y_i - X_i'\beta)^2 = (X'X)^{-1} X'Y$$

REMEMBER

If *Orthogonality Condition* is broken, i.e.

biased and inconsistent.

How frequently does it happen ?

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REMEMBER $\hat\beta^{(OLS,n)}$ is solution of normal equations $X\left(Y - X\beta\right) = 0$.

If *Orthogonality Condition* is broken, i. e.

$$\text{plim}_{T \to \infty} \frac{1}{n}\sum_{i=1}^n X_i \varepsilon_i = E\left(X_1 \varepsilon_1\right) \neq 0$$

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EXAMPLES of SITUATIONS WHEN EXPLANATORY VARIABLES AND DISTURBANCES ARE CORRELATED

General examples:

1. Measurement of explanatory variable with a random error,
2. lagged values of response variable serve as explanatory ones,
3. system of regression equations (SUE, SE).

Specific examples:

1. Consumption always depends on the income of households,
2. inflation typically depends on interest rate, etc.

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where

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which were found as *"substitutes" (instruments)* for X_i, so that

$$\underset{T \to \infty}{\text{plim}} \; \frac{1}{n} \sum_{i=1}^{n} Z_i \varepsilon_i = E\left\{Z_1 \varepsilon_1\right\} = 0.$$

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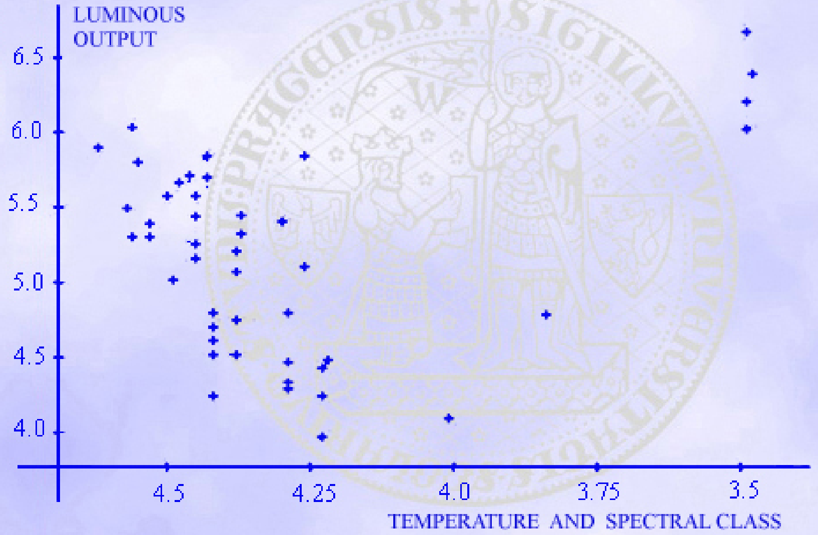
$$\hat{\beta}^{(IV,n)} = (Z'X)^{-1} Z'Y = \beta^0 + \left(\frac{1}{T}\sum_{i=1}^{n} Z_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} Z_i \varepsilon_i$$

Content of contribution


LUMINOUS OUTPUT
6.5
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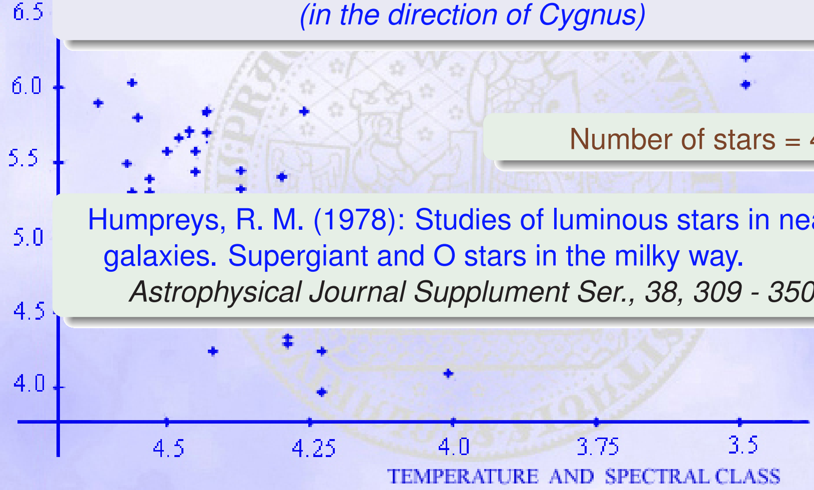
TEMPERATURE AND SPECTRAL CLASS

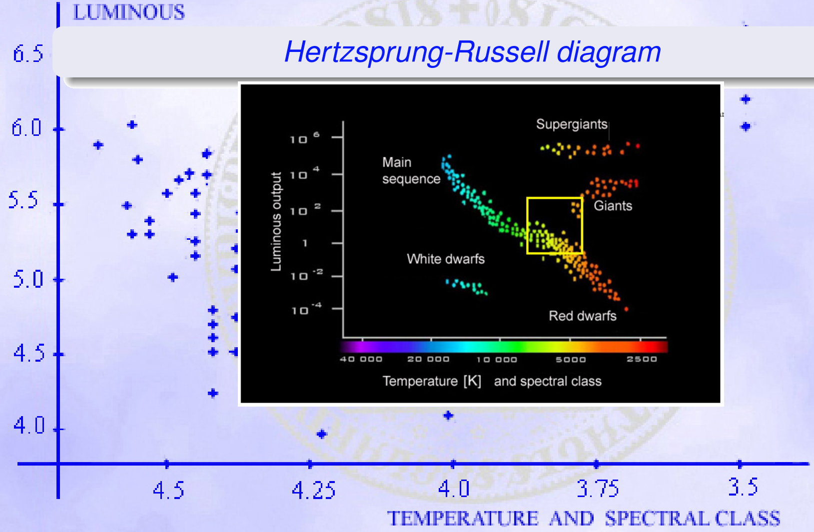
Hertzsprung-Russell diagram of stars cluster CYG OB1
(in the direction of Cygnus)



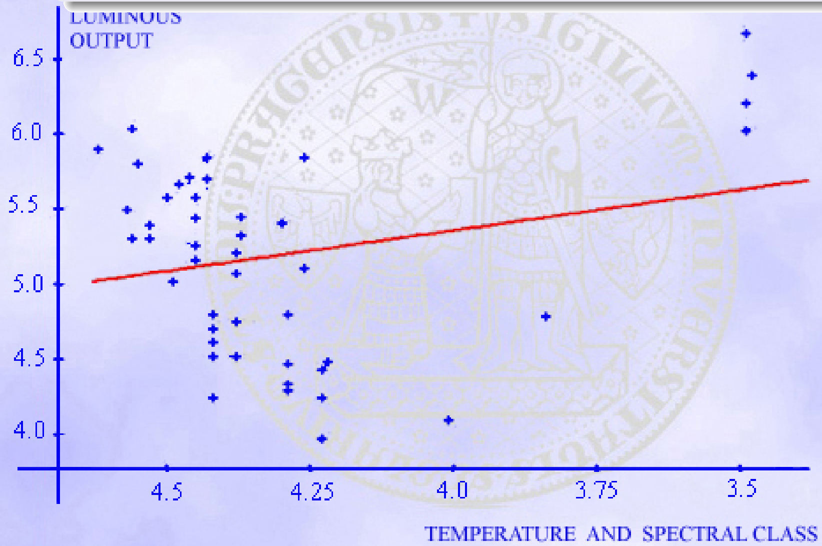
Number of stars = 47

Humpreys, R. M. (1978): Studies of luminous stars in nearby galaxies. Supergiant and O stars in the milky way. *Astrophysical Journal Supplument Ser., 38, 309 - 350.*

Hertzsprung-Russell diagram



The ordinary least squares

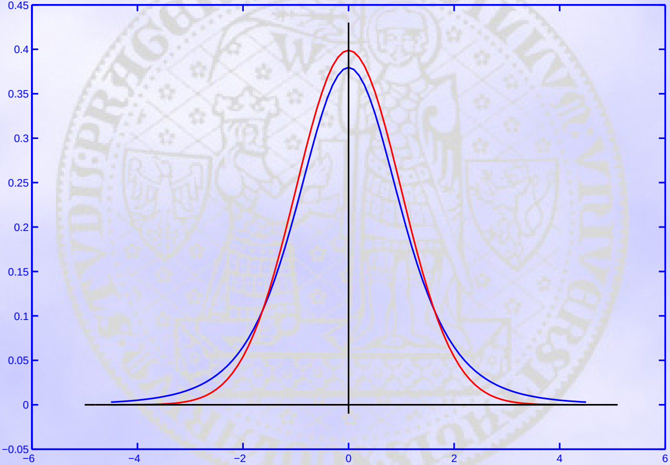


Fisher, R. A. (1922): On the mathematical foundation of theoretical statistics. *Philos. Trans. Roy. Soc. London Ser. A 222, 309 - 368.*

$$s_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

Degrees of freedom	t_9	t_5	t_3
$\frac{\mathrm{var}_{N(0,1)}(s_n^2)}{\mathrm{var}_{t(\nu)}(s_n^2)}$	0.83	0.40	0!

The blue curve is standard normal while the red one is the Student's with 5 degrees of freedom.



Content of contribution

Method of *the least weighted squares* (*LWS*)

Residuals

$$r_i^2(\beta) = \left(Y_i - \sum_{j=1}^{p} X_{ij}\beta_j\right)^2$$

Order statistics of squared residuals, i. e.

$$r_{(1)}^2(\beta) \le r_{(2)}^2(\beta) \le \ldots \le r_{(n)}^2(\beta)$$

Weights

$$1 \ \ge \ w_1 \ \ge \ w_2 \ \ge \ \ldots \ \ge \ w_n \ \ge \ 0$$

$$\hat{\beta}^{(LWS,n,h)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} w_i \cdot r_{(i)}^2(\beta)$$

Method of *the least weighted squares* (*LWS*)

Order statistics of squared residuals, i. e.

$$r^2_{(1)}(\beta) \leq r^2_{(2)}(\beta) \leq \ldots \leq r^2_{(n)}(\beta),$$

weight function

$$w(s) : [0,1] \rightarrow [0,1], \; w(1) = 1, \text{ nonincreasing}$$

$$\hat{\beta}^{(LWS,n,h)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} w\left(\frac{i-1}{n}\right) r^2_{(i)}(\beta)$$

Method

Order st

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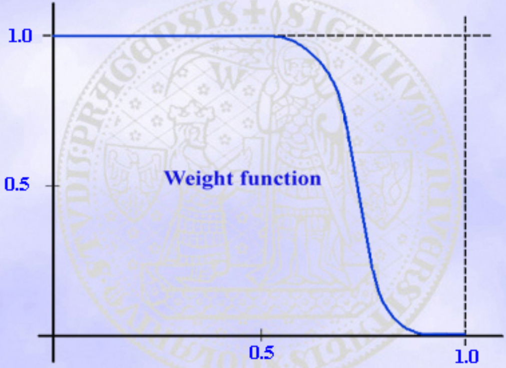

$w_i = w\left(\frac{i-1}{n}\right)$

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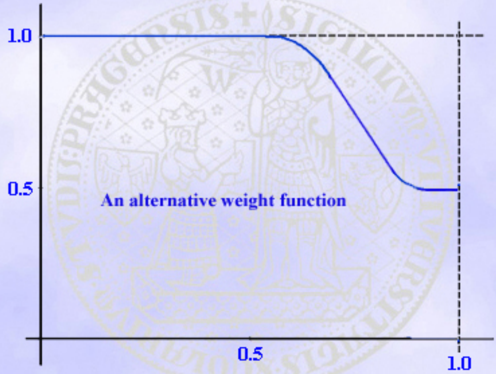

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Víšek, J. Á. (2000): Regression with high breakdown point. *Robust 2000 (eds. Antoch, J. Dohnal, G.), 324 - 356.*

Method of *the least weighted squares* (*LWS*)

Ranks of the squared residuals

$$\pi(\beta, j) = i \in \{1, 2, ..., n\} \qquad \Leftrightarrow \qquad r_j^2(\beta) = r_{(i)}^2(\beta)$$

$$\hat{\beta}^{(LWS,n,h)} = \arg\min_{\beta \in R^p} \sum_{i=1}^n w\left(\frac{i-1}{n}\right) r_{(i)}^2(\beta)$$

$$= \arg\min_{\beta \in R^p} \sum_{j=1}^n w\left(\frac{\pi(\beta,j)-1}{n}\right) r_j^2(\beta)$$

Normal equations for the least weighted squares

$$\sum_{j=1}^n w\left(\frac{\pi(\beta,j)-1}{n}\right) X_j \left(Y_j - X_j'\beta\right) = 0.$$

Let's recall:

Normal equations for the ordinary least squares

$$\sum_{j=1}^{n} X_j \left(Y_j - X_j' \beta \right) = 0$$

and compare it with:

Normal equations for the least weighted squares

$$\sum_{j=1}^{n} w \left(\frac{\pi(\beta, j) - 1}{n} \right) X_j \left(Y_j - X_j' \beta \right) = 0.$$

Method of the instrumental variables - theory

Estimate by the method of the instrumental variables is given by

$$\hat{\beta}^{(IV,n)} = \beta^0 + \left(\frac{1}{n}\sum_{j=1}^n Z_j X_j'\right)^{-1} \frac{1}{n}\sum_{j=1}^n Z_j \varepsilon_j,$$

$$\sum_{j=1}^n Z_j \left(Y_j - X_j'\beta\right) = 0$$

i. e., the estimate is *unbiased and consistent*.

Unfortunately, it is not *robust*.

Robustification is straightforward !

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$$\sum_{j=1}^{n} w\left(\frac{\pi(\beta,j)-1}{n}\right) Z_j \left(Y_j - X_j'\beta\right) = 0$$

Instrumental weighted variables (*IWV*) - definition

Definition

The estimator by means of *the instrumental weighted variables* $\hat{\beta}^{(IWV,n,w)}$ is defined as (any) solution of *the normal equations*

$$\sum_{j=1}^{n} w\left(\frac{\pi(\beta, j) - 1}{n}\right) Z_j \left(Y_j - X_j'\beta\right) = 0.$$

Víšek, J. Á. (2004): Robustifying instrumental variables. *Proceedings of COMPSTAT'2004, Physica-Verlag/Springer, 1947 - 1954.*

Instrumental weighted variables (*IWV*) - algorithm

Víšek, J. Á. (2006): Instrumental Weighted Variables - algorithm. *Proceedings of COMPSTAT'2006, Physica-Verlag/Springer, 777-786.*

Instrumental weighted variables - asymptotic theory

Theorem

Let **C1, C2, C3** and **C4** hold. Then the estimator by *instrumetal weighted variables-* $\hat{\beta}^{(IWV,n,w)}$ is consistent, i. e.

$$\hat{\beta}^{(IWV,n,w)} \underset{\mathbf{p}}{\longrightarrow} \beta^0 \quad \text{for} \quad n \to \infty.$$

Víšek, J. Á. (2009): Consistency of the instrumental weighted variables. *Annals of the Institute of Statistical Mathematics, Vol.61, No.3 (September, 2009).*

Instrumental weighted variables - asymptotic theory

Conditions

NC1 *Random variables*

- ✓ $\left\{(X_i', Z_i', \varepsilon_i)'\right\}_{i=1}^{\infty}$ - sequence independent equally distributed r.v.'s,
- ✓ $\forall \ (i \in N) \quad Z_i$ and ε_i - mutually independent,
- ✓ D.f. $F_{X,Z}(x, z)$ - absolutely continuous,
- ✓ $E\left\{w\left(F_{\beta^0}(|\varepsilon_1|)\right) Z_1 X_1'\right\}$ a $E\left\{Z_1 Z_1'\right\}$ - positive definite,
- ✓ $\exists \ (q > 1) \ : \ E\left\{\|Z_1\| \cdot \|X_1\|\right\}^q < \infty$,
- density $f_{e|X}(r|X_1 = x)$ is uniformly in x Lipschitz of the first order,
- $|f_e'(r)| < U < \infty$.

Instrumental weighted variables - asymptotic theory

NC2 *Weight function*

Conditions

- ✓ $w(\alpha) : [0,1] \rightarrow [0,1], w(0) = 1,$
- ✓ absolutely continuous, nonincreasing,
- ✓ $\exists$ derivative $w'(\alpha) > -L, \ L \in R^+,$
- $w'(\alpha)$ is Lipschitz of the first order.

Theorem

Let **NC1, NC2, C3** a **C4** hold. Then the estimator by means of *the instrumental weighted variables* is $\sqrt{n}$-consistent, i. e.

$$\forall(\varepsilon > 0) \ \exists(K_\varepsilon \in R, n_\varepsilon \in N) \ \forall(n > n_\varepsilon)$$

$$P\left(\left\|\sqrt{n}\left(\hat{\beta}^{(IWV,n)} - \beta^0\right)\right\| < K_\varepsilon\right) > 1 - \varepsilon.$$

Key result!

Instrumental weighted variables - asymptotic theory

Let's denote $g(r)$ density of r. v. e_1^2.

Conditions

AC1 D. f. of error term

- $\forall (a \in R^+) \ \exists \ \left(\Delta(a) > 0\right) \quad \inf_{r \in (0, a+\Delta(a))} g(r) > L_g > 0$
- $\exists (s > 1) \ : \ E \left|\varepsilon_1\right|^{2s} < \infty$

Theorem

Let $Q = E\left\{w\left(F_{\beta^0}(|\varepsilon_1|)\right) Z_1 X_1'\right\}$ and let **NC1, NC2, C3, C4** and **AC1** be fulfilled. Then for the estimator by means of the *instrumental weighted variables* we have following Bahadur representation

$$\sqrt{n}\left(\hat{\beta}^{(IWV,n,w)} - \beta^0\right) = Q^{-1} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w\left(F_{\beta^0}(|\varepsilon_i|)\right) \cdot Z_i \varepsilon_i + o_p(1)$$

for $n \to \infty$.

THE LEAST SQUARES

$$\sqrt{n}\left(\hat{\beta}^{(LS,n)} - \beta^0\right) = \left(\frac{1}{n}X'X\right)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i\varepsilon_i$$

THE INSTRUMENTAL WEIGHTED VARIABLES

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The least squares

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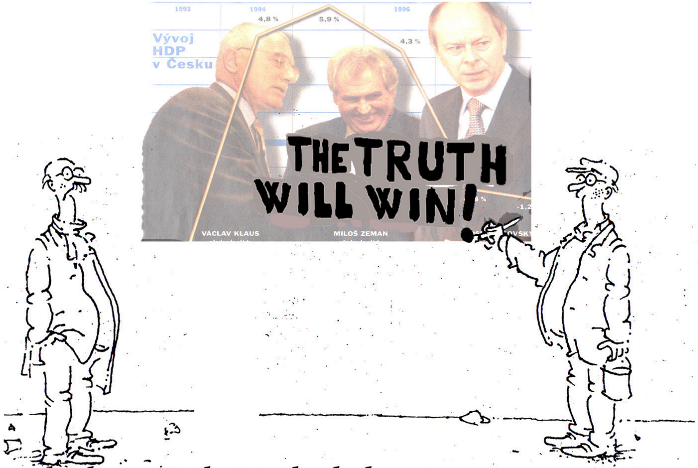
$$r\left(\hat{\beta}^{(LS,n)}\right) = \left(I - \frac{1}{n}X\left(\frac{1}{n}X'X\right)^{-1} X'\right)\varepsilon$$

The instrumental weighted variables

$$\sqrt{n}\left(\hat{\beta}^{(IWV,n,w)} - \beta^0\right) = Q^{-1} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w\left(F_{\beta^0}(|\varepsilon_i|)\right) \cdot Z_i \varepsilon_i + o_p(1)$$

$$r\left(\hat{\beta}^{(IWV,n,w)}\right) = \left(I - \frac{1}{n}XQ^{-1}X'\right)\varepsilon + o_p(n^{-\frac{1}{2}})$$

JEN ABY VÁS NĚKDO NEPROHNAL ZA ŠÍŘENÍ POPLAŠNÉ ZPRÁVY!



Just make sure that nobody boxes your ears for spreading misleading information.

THANKS FOR ATTENTION

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