Segmented Regression Based on Cut-off Polynomials

Miloš Kaňka[1] | *University of Economics, Prague, Czech Republic*

Abstract

In *Statistika: Statistics and Economy Journal* No. 4/2015 (pp. 39–58), author's paper *Segmented Regression Based on B-splines with Solved Examples* was published. Use of B-spline basis functions has many advantages, the most important being a special form of matrix of system of normal equations suitable for quick solution of this system.

The subject of this paper is to explain how that segmented regression can be mathematically developed in other way, which doesn't require the knowledge of relatively complicated theory of B-spline basis functions, but is based on simpler apparatus of cut-off polynomials. The author considers a detailed calculation of matrix of system of normal equations elements and elaboration of so called polygonal method, as his contribution to issues of segmented regression. This method can be used to automatically obtain required values of nodal points. Author pays major attention to computing elements of matrix of system of normal equations, which he also developed as computer program called TRIO.

Keywords	***JEL code***
Normal equations, polygonal method, cut-off polynomials, linear, quadratic and cubic segmented regression, transformation of the parametric variable	*C10, C63, C65*

INTRODUCTION

This work deals with segmented regression based on splines as cut-off polynomials in three particular cases, so-called linear, quadratic, and cubic (segmented) regression. We introduce the so-called polynomial method of parametric-variable value assignment to experimentally obtained points (which lie generally in $\mathbb{R}^m$ of integer dimension $m \geq 1$), augmented with the computation of the so-called knot values of the variable corresponding to the division of these points into groups (segments).

To improve the numerical stability of parametric equations of the regression curves, we describe a transformation of the initial interval into a unit interval. Lastly, we choose the optimal regression curve for a given problem according to the measure of the determination indices of the three regression cases mentioned above.

Segmented regression can also be based on so-called B-spline functions. Through this method the matrix of the system of normal equations tridiagonal (for linear regression), five-diagonal (for quadratic regression), or seven-diagonal (for cubic regression) that enables a less elaborate solution to the given system, can be applied, see Kaňka (2015).

[1] Ekonomická 957, 148 00 Prague, Czech Republic. E-mail: kanka@vse.cz.

1 NORMAL EQUATIONS

Let $n \geq 2$ be an integer, in the Euclidean space $\mathbb{R}^m$ ($m > 1$ integer) let us consider n points $P_i = x_j^{(i)}$ ($i = 1, ..., n; j = 1, ..., m$), not necessarily distinct (except for the case when all would be equal). Besides these points, for which we assume that $x_j^{(i)}$ are real random variables depending on a real variable t e.g. the time, we consider the so-called knots $T_0 < T_1 < ... < T_k$, where $k \geq 1$ is an integer, the so-called complementary knots $T_0 < T_1$ and $T_{k+1} > T_k$. The points $T_1 < T_2 < ... < T_k$ are called main knots.

In the intervals (T_{l-1}, T_l), $l = 1, ..., k + 1$, where the variable t changes, let us consider the increasing sequence $t_{l,1} < t_{l,2} < ... < t_{l,n(l)}, n(l) \geq 1$ integer, and let to every such member correspond exactly one point $x_j^{(lw)}$, $w = 1, ..., n(l)$. It holds then that $n = \sum_{l=1}^{k+1} n(l)$. The knots form the boundaries of the intervals, in the union of which we shall consider, depending on the chosen number $Q \in \{1, 2, 3\}$, the following real-valued functions of the real variable t

$$g_j(t) = \gamma_j^{(1)} + \gamma_j^{(2)} t + \cdots + \gamma_j^{(Q+1)} t^Q + \sum_{r=1}^{k} \gamma_j^{(r+Q+1)} [(t - T_r)_+]^Q, \tag{1}$$

where $\gamma_j^{(1)}, \gamma_j^{(2)}, ..., \gamma_j^{(k+Q+1)}$ are real parameters, i.e., linear (for $Q = 1$), quadratic (for $Q = 2$), and cubic (for $Q = 3$) splines in the form of so-called cut-off polynomials. By the symbol $(x)_+$ we will understand the following real-valued function of a real variable:

$$(x)_+ = \begin{cases} x, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

Instead of $[(t - T_r)_+]^Q$, we shall write in short $(t - T_r)_+^Q$.

Example 1.1

Let $k = 2, Q = 3, T_0 < T_1 < T_2 < T_3$, where $T_0 = 0, T_1 = 1, T_2 = 3, T_3 = 5$. There is (see formula 1):

$$g_j(t) = \gamma_j^{(1)} + \gamma_j^{(2)} t + \gamma_j^{(3)} t^2 + \gamma_j^{(4)} t^3 + \gamma_j^{(5)} (t-1)_+^3 + \gamma_j^{(6)} (t-3)_+^3,$$

thus (if we denote $\gamma_j^{(1)} + \gamma_j^{(2)} t + \gamma_j^{(3)} t^2 + \gamma_j^{(4)} t^3 = h_j(t)$)

$$g_j(t) = \begin{cases} h_j(t) & \text{for } 0 \leq t < 1, \\ h_j(t) + \gamma_j^{(5)} (t-1)^3 & \text{for } 1 \leq t < 3, \\ h_j(t) + \gamma_j^{(5)} (t-1)^3 + \gamma_j^{(6)} (t-3)^3 & \text{for } 3 \leq t \leq 5. \end{cases}$$

We shall assume that the observed process is additive, i.e., for every possible value j, l, w the following holds:

$$x_j^{(lw)} = g_j(t_{lw}) + \epsilon_j^{(lw)},$$

where $\epsilon_j^{(lw)}$ are identically distributed random variables with the constant variance. Then we may obtain the estimates $c_j^{(1)}, c_j^{(2)}, \dots, c_j^{(k+Q+1)}$ of the parameters $\gamma_j^{(1)}, \gamma_j^{(2)}, \dots, \gamma_j^{(k+Q+1)}$ by the least squares method:

$$U_j = \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} [x_j^{(lw)} - g_j(t_{lw})]^2 =$$

$$= \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} \left[-x_j^{(lw)} + \sum_{q=1}^{Q+1} \gamma_j^{(q)} t_{lw}^{q-1} + \sum_{r=1}^{k} \gamma_j^{(r+Q+1)} (t_{lw} - T_r)_+^Q \right]^2.$$

By partial derivation we get for $1 \leq p \leq Q+1$ that:

$$\frac{\partial U_j}{\partial \gamma_j^{(p)}} = 2\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}\left[-x_j^{(lw)} + \sum_{q=1}^{Q+1}\gamma_j^{(q)} t_{lw}^{q-1} + \sum_{r=1}^{k}\gamma_j^{(r+Q+1)}(t_{lw}-T_r)_+^Q\right]\cdot t_{lw}^{p-1}, \tag{2}$$

for $Q+1 < p \leq k+Q+1$

$$\frac{\partial U_j}{\partial \gamma_j^{(p)}} = 2\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}\left[-x_j^{(lw)} + \sum_{q=1}^{Q+1}\gamma_j^{(q)} t_{lw}^{q-1} + \sum_{r=1}^{k}\gamma_j^{(r+Q+1)}(t_{lw}-T_r)_+^Q\right]\cdot \\ \cdot\left(t_{lw}-T_{p-(Q+1)}\right)_+^Q, \tag{3}$$

(when we put $\mathbf{r}+Q+1=p$, then $\mathbf{r}=p-(Q+1)$).

Case A: $1 \leq p \leq Q+1$ shall lead to (see formula 2, without the multiplier 2 on the right side of the equation):

$$\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}\left[-x_j^{(lw)} + \sum_{q=1}^{Q+1}\gamma_j^{(q)} t_{lw}^{q-1} + \sum_{r=1}^{k}\gamma_j^{(r+Q+1)}(t_{lw}-T_r)_+^Q\right]\cdot t_{lw}^{p-1}$$
$$= \sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p-1}\left[\sum_{q=1}^{Q+1}\gamma_j^{(q)} t_{lw}^{q-1} + \sum_{r=1}^{k}\gamma_j^{(r+Q+1)}(t_{lw}-T_r)_+^Q - x_j^{(lw)}\right].$$

In the second sum in the square brackets we will change the summation index from $r: 1 \rightarrow k$ to $r: 1+(Q+1) \rightarrow k+(Q+1)$ with proper adjustments to the summed expression:

$$\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p-1}\left[\sum_{q=1}^{Q+1}\gamma_j^{(q)} t_{lw}^{q-1} + \sum_{r=1+(Q+1)}^{k+(Q+1)}\gamma_j^{(r)}\left(t_{lw}-T_{r-(Q+1)}\right)_+^Q - x_j^{(lw)}\right] =$$

$$= \sum_{q=1}^{Q+1}\gamma_j^{(q)}\underbrace{\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p+q-2}}_{m_{pq}} + \sum_{r=Q+2}^{k+Q+1}\gamma_j^{(r)}\underbrace{\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p-1}\left(t_{lw}-T_{r-(Q+1)}\right)_+^Q}_{m_{pr}} - \underbrace{\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} x_j^{(lw)} t_{lw}^{p-1}}_{z_{pj}},$$

where:

$$m_{pq} = \begin{cases} \sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p+q-2} & \text{for } A_1: 1 \leq q \leq Q+1, \\ \sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p-1}(t_{lw}-T_{q-(Q+1)})_+^Q & \text{for } A_2: Q+1 < q \leq k+Q+1, \end{cases} \tag{4}$$

and

$$z_{pj} = \sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} x_j^{(lw)} t_{lw}^{p-1}. \tag{5}$$

Case B: $Q+1<p\leq k+Q+1$ shall analogously lead to (see formula 3):

$$\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}\left[-x_j^{(lw)}+\sum_{q=1}^{Q+1}\gamma_j^{(q)}t_{lw}^{q-1}+\sum_{r=1}^{k}\gamma_j^{(r+Q+1)}(t_{lw}-T_r)_+^Q\right]\cdot\left(t_{lw}-T_{p-(Q+1)}\right)_+^Q=$$
$$=\sum_{q=1}^{Q+1}m_{pq}\gamma_j^{(q)}+\sum_{r=Q+2}^{k+Q+1}m_{pr}\gamma_j^{(r)}-z_{pj},$$

where:

$$m_{pq}=$$
$$=\begin{cases}\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}t_{lw}^{q-1}(t_{lw}-T_{p-(Q+1)})_+^Q & \text{for } B_1: 1\leq q\leq Q+1,\\ \sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}(t_{lw}-T_{q-(Q+1)})_+^Q(t_{lw}-T_{p-(Q+1)})_+^Q & \text{for } B_2: Q+1<q\leq k+Q+1,\end{cases} \tag{6}$$

and

$$z_{pj}=\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}x_j^{(lw)}\left(t_{lw}-T_{p-(Q+1)}\right)_+^Q. \tag{7}$$

Example 1.2

Let $k=2$, $Q=3$, hence $k+Q+1=2+3+1=6$. We are to determine m_{pq} for $p=3$, $q=6$, thus $m_{36,}$ and also $z_{pj}=z_{3j}$. It corresponds to Case A together with A_2 hence, according to (4):

$$m_{36}=\sum_{l=1}^{3}\sum_{w=1}^{n(l)}t_{lw}^2(t_{lw}-T_2)_+^3$$

and, according to (5):

$$z_{3j}=\sum_{l=1}^{3}\sum_{w=1}^{n(l)}x_j^{(lw)}t_{lw}^2.$$

Further, we are to determine m_{pq} for $p=6$, $q=3$, that is m_{63}, and also $z_{pj}=z_{6j}$. It corresponds to Case B together with B_1 hence, according to (6):

$$m_{63}=\sum_{l=1}^{3}\sum_{w=1}^{n(l)}t_{lw}^2(t_{lw}-T_2)_+^3$$

and, according to (7):

$$z_{6j}=\sum_{l=1}^{3}\sum_{w=1}^{n(l)}x_j^{(lw)}(t_{lw}-T_2)_+^3.$$

It holds that $m_{36}=m_{63}$.

Furthermore, we are to determine m_{pq} for $p = q = 5$, that is m_{55}, and also $z_{pj} = z_{5j}$. It corresponds to Case B together with B_2 hence, according to (6):

$$m_{55} = \sum_{l=1}^{3} \sum_{w=1}^{n(l)} (t_{lw} - T_1)_+^3 (t_{lw} - T_1)_+^3$$

and, according to (7):

$$z_{3j} = \sum_{l=1}^{3} \sum_{w=1}^{n(l)} x_j^{(lw)} t_{lw}^2.$$

Lastly, we are to determine m_{11} It corresponds to Case A together with A_1 hence, according to (4):

$$m_{11} = \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} t_{lw}^0 = \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} 1 = \sum_{l=1}^{k+1} n(l) = n.$$

This result holds for $k \geq 1$ arbitrary, independently of $Q \in \{1, 2, 3\}$.

We proceed in the minimization of U_j It is known from the general theory of mathematical analysis that a necessary condition for U_j as a function of the parameters $\gamma_j^{(1)}, \gamma_j^{(2)}, \dots, \gamma_j^{(k+Q+1)}$, to attain its minimum is the system of equations $\frac{\partial U_j}{\partial \gamma_j^{(p)}} = 0$ for $p = 1, \dots, k + Q + 1$ (see (2), (3)). Based on the aforementioned results (see Cases A and B), we arrive at the system of $k + Q + 1$ linear equations for the estimates $c_j^{(1)}, c_j^{(2)}, \dots, c_j^{(k+Q+1)}$:

$$\boldsymbol{M}\boldsymbol{c}_j = \boldsymbol{Z}_j, \tag{8}$$

where: $\boldsymbol{M} = (m_{pq})_{1 \leq p,q \leq k+Q+1}$ is a $(k + Q + 1) \times (k + Q + 1)$ matrix, and $\boldsymbol{Z}_j = (z_{pj})_{1 \leq p \leq k+Q+1}$ and $\boldsymbol{c}_j = \left(c_j^{(p)}\right)_{1 \leq p \leq k+Q+1}$ are $(k + Q + 1) \times 1$ matrices. The equations of the system (8) are called normal equations. We can easily verify that $\boldsymbol{M}$ is a symmetric matrix.

By solving the system of equations (8) we get the sought estimates $c_j^{(1)}, c_j^{(2)}, \dots, c_j^{(k+Q+1)}$ of the parameters $\gamma_j^{(1)}, \gamma_j^{(2)}, \dots, \gamma_j^{(k+Q+1)}$ in the linear combination $g_j(t)$, see (1), $\in \langle T_0, T_{k+1} \rangle$. To these estimates we get the corresponding regression splines (linear for Q = 1, quadratic for Q = 2, and cubic for Q = 3) for $t \in \langle T_0, T_{k+1} \rangle$ through the equation:

$$x_j = G_j(t) = c_j^{(1)} + c_j^{(2)} t + \dots + c_j^{(Q+1)} t^Q + \sum_{r=1}^{k} c_j^{(r+Q+1)} (t - T_r)_+^Q. \tag{9}$$

To sum up, (for $j = 1, \dots, m$) these equations represent the parametric expression of a curve in $\mathbb{R}^m = (0; x_1, x_2, \dots, x_m)$ that is the output of the regression model for the observed process. In short, we shall call it a regression curve (linear for Q = 1, quadratic for Q = 2, and cubic for = 3).[2]

[2] Literature: Spline functions – Bézier (1972), Böhmer (1974), Kaňka (2015), Makarov, Chlobystov (1983), Schmitter, Delgado-Gonzalo, Unser (2016), Sung (2016), Spät (1996), Vasilenko (1983), Wang, Wu, Shen, Zhang, Mou, Zhou (2016); Cut-off polynomials – Meloun, Militký (1994); Segmented regression – Feder (1975), Guzik (1974), Seger (1988).

2 POLYGONAL METHOD

For greater clarity, we shall confine ourselves to the plane $\mathbb{R}^2$.

In $\mathbb{R}^2$ let us consider the connected oriented graph $\vec{G} = [A, \vec{B}]$, with the set of vertices $A = \{1,2,3, ..., n\}$, $n \geq 2$, $\vec{B} = \{(1,2),(2,3), ..., (n-1,n)\}$ is the set of the (oriented) edges. We can imagine that the planar polygonal trail created like this and having its initial point in 1 and end point in n represents an idealized route of a car moving at a constant speed that started from place 1 and ended the journey at place n. Each vertex of the graph can be regarded as an experimental point, the coordinates of which are obtained by measuring its distance (e.g. in km) from the left and bottom edge of the map. We divide the graph vertices into $k + 1$ groups ($k \geq 1$) by $n(l) \geq 1$ points $x_j^{(lw)}$ ($l = 1, ..., k + 1$; $w = 1, ..., n(l)$; $j = 1,2$) in such a way that it holds that:

$$n = \sum_{l=1}^{k+1} n(l)$$

(such a division might be caused, e.g., by the difficulty of the corresponding road terrain), and assign to them (increasing) values t_{lw} (in km), where t_{lw} denotes the length of the passed route from the start 1 to the place $x_j^{(lw)}$, which could be regarded as a spot for a short break.

For $l = 1, ..., k + 1$ we place the values $t_{l1} < t_{l2} < ... < t_{l,n(l)}$ into the interval $\langle T_{l-1}, T_l)$. Meanwhile, we will demand that the sequence of points $T_0, T_1, ..., T_k, T_{k+1}$ is increasing and it holds that $T_{l-1} \leq t_{l1}$ for $l = 1, ..., k + 1$($T_1, ..., T_k$ are called main and the points $T_0 < T_1$ and $T_k + 1 > T_k$ complementary knots for the observed drive; compare with Section 1).

It gives sense to set $T_0 = 0$ then further it follows from the relation $T_l \leq t_{l+1,1}$ after substituting for $T_l = \lfloor t_{l,n(l)} \rfloor + p_l$ the inequality $\lfloor t_{l,n(l)} \rfloor + p_l \leq t_{l+1,1}$, thus $p_l \leq t_{l+1,1} - \lfloor t_{l,n(l)} \rfloor$. Let:

$$P = \min_{l=1,\ldots,k} \{t_{l+1,1} - \lfloor t_{l,n(l)} \rfloor\} \tag{10}$$

and $p =$ P (by the symbol P we shall understand the integer part of the corresponding real number). If $p \geq 1$, then we set $p_l = p$; we shall return to the case $p = 0$.

If we disregard the car drive example, we can say that there exists a certain automation for "operating" variable value assignment to experimentally obtained points divided in a certain way into groups, followed by the computation of so-called knots which define the interval of the assigned variable to the given group. We shall call this automation in short the "polygonal method". This procedure is implemented in the computer program TRIO, which is able to solve regression problems for $Q \in \{1, 2, 3\}$,in the plane $\mathbb{R}^2$ and in the space $\mathbb{R}^3$ as well.

Example 2.1

Let us consider in $\mathbb{R}^3$ the points $x_j^{(lw)}$ ($l = 1,2,3$; $w = 1, ..., n(l)$, where $n(1) = 2$, $n(2) = 3$, $n(3) = 2$; $j = 1,2,3$), divided into three groups:

$x_j^{(11)} = (1, 1, 1),$ $x_j^{(12)} = (1.1, 1.2, 1.3),$

$x_j^{(21)} = (1.5, 1, 1.4),$ $x_j^{(22)} = (2, 3, 4),$ $x_j^{(23)} = (3, 3, 5),$

$x_j^{(31)} = (3.1, 3.2, 5.05),$ $x_j^{(32)} = (4, 4, 6).$

According to the polygonal method, we assign to them (increasing) values of an operating variable (for example, time):

$$t_{11} = 0.0000, \qquad t_{12} = 0.3742,$$

$$t_{21} = 0.8325, \qquad t_{22} = 4.1506, \qquad t_{23} = 5.5648,$$

$$t_{31} = 5.7939, \qquad t_{32} = 7.3277.$$

According to (10), there is $P = \min\{0.8325, 0.7939\} = 0.7939$, hence $p = \lfloor P \rfloor = 0$.

We proceed further as follows. We substitute $x_j^{(lw)}$ with points $\tilde{x}_j^{(lw)} = L \cdot x_j^{(lw)}$, where L > 1 is sufficiently large, and assign to them through the polygonal method (increasing) values of an operating variable $\tilde{t}_{lw} = 10 \cdot t_{lw}$. In this way we obtain $P = \min\{5.3243, 2.9390\}$, thus $p = \lfloor P \rfloor = 2$. We get the knots $\tilde{T}_0 = 0, \tilde{T}_1 = 5, \tilde{T}_2 = 57, \tilde{T}_3 = 75$, which correspond to the initial knots $T_0 = 0, T_l = \frac{\tilde{T}_l}{L} (L = 10)$, that is, the points $T_0, T_1 = 0.5, T_2 = 5.7, T_3 = 7.5$, through which we carry out the sought segmented regression. The program TRIO has this procedure built in.

3 TRANSFORMATION OF THE PARAMETRIC VARIABLE

The elements of the matrix $\boldsymbol{M}$ and the vector Z_j of the system of normal equations (see (8)) are structured in such a way that reflects the fact that we are working with splines based on cut-off polynomials. To increase the numerical stability of the parametric equations of the regression curve (see (9)), which is the output of the given regression mode, the scientific literature proposes to implement a transformation of the corresponding parameter into a, for example, unit-length interval (if the length of the initial interval is greater than 1).

Let us deal with such a transformation in the case of, for example, quadratic regression, i.e., when $Q = 2$. Let us write the matrix of the system of normal equations $\boldsymbol{M} = (m_{pq})_{1 \le p,q \le k+3}$ in the following form:

$$\boldsymbol{M} = \begin{pmatrix} n & m_{12} & \boldsymbol{A} \\ m_{12} & m_{22} & \boldsymbol{B} \\ \boldsymbol{A}^T & \boldsymbol{B}^T & \boldsymbol{D} \end{pmatrix},$$

where:

$$n = m_{11},$$

$$\boldsymbol{A} = (m_{13}, m_{14}, \dots, m_{1,k+3}),$$

$$\boldsymbol{B} = (m_{23}, m_{,24}, \dots, m_{2,k+3}),$$

$\boldsymbol{A}^T$ is the transpose matrix of $\boldsymbol{A}$,

$\boldsymbol{B}^T$ is the transpose matrix of $\boldsymbol{B}$,

$$\boldsymbol{D} = \begin{pmatrix} m_{33} & m_{34} & \cdots & m_{3,k+3} \\ m_{34} & m_{44} & \cdots & m_{4,k+3} \\ \vdots & \vdots & \ddots & \vdots \\ m_{3,k+3} & m_{4,k+3} & \cdots & m_{k+3,k+3} \end{pmatrix},$$

and let us write the $k + 3$-dimensional vector $\boldsymbol{Z}_j = (z_{pj})_{1 \le p \le k+3}$ as:

$$\boldsymbol{Z}_j = \big(z_{1j}, z_{2j}, (z_{3j}, z_{4j}, \dots, z_{k+3,j})\big) = (z_{1j}, z_{2j}, \boldsymbol{E}_j).$$

If we subject every value t_{lw} ($l = 1, ..., k + 1$; $w = 1, ..., n(l)$), including the knots $T_0, T_1, ..., T_k, T_{k+1}$, to the transformation

$$t'_{lw} = Kt_{lw},$$
$$T'_0 = KT_0, \qquad T'_l = KT_l, \tag{11}$$

where $K = 1/(T_{k+1} - T_0)$, the matrix $\boldsymbol{M}$ transforms into:

$$\boldsymbol{M}' = \begin{pmatrix} n & Km_{12} & K^2\boldsymbol{A} \\ Km_{12} & K^2m_{22} & K^3\boldsymbol{B} \\ K^2\boldsymbol{A}^T & K^3\boldsymbol{B}^T & K^4\boldsymbol{D} \end{pmatrix}$$

and $\boldsymbol{Z}_j$ transforms into $\boldsymbol{Z}'_j = (z_{1j}, Kz_{2j}, K^2\boldsymbol{E}_j)$. Now, if the (k + 3) – dimensional vector:

$$\boldsymbol{c}_j = \left(c_j^{(p)}\right) = \left(c_j^{(1)}, c_j^{(2)}, c_j^{(3)}, c_j^{(4)}, \dots, c_j^{(k+3)}\right) = \left(c_j^{(1)}, c_j^{(2)}, \boldsymbol{C}_j\right)$$

is the solution of the system of normal equations, the following equality holds:

$$\begin{pmatrix} n & m_{12} & \boldsymbol{A} \\ m_{12} & m_{22} & \boldsymbol{B} \\ \boldsymbol{A}^T & \boldsymbol{B}^T & \boldsymbol{D} \end{pmatrix} \begin{pmatrix} c_j^{(1)} \\ c_j^{(2)} \\ \boldsymbol{C}_j^T \end{pmatrix} = \begin{pmatrix} z_{1j} \\ z_{2j} \\ \boldsymbol{E}_j^T \end{pmatrix},$$

which is equivalent to the equality:

$$\begin{pmatrix} n & Km_{12} & K^2\boldsymbol{A} \\ Km_{12} & K^2m_{22} & K^3\boldsymbol{B} \\ K^2\boldsymbol{A}^T & K^3\boldsymbol{B}^T & K^4\boldsymbol{D} \end{pmatrix} \begin{pmatrix} c_j^{(1)} \\ c_j^{(2)}/K \\ \boldsymbol{C}_j^T/K^2 \end{pmatrix} = \begin{pmatrix} z_{1j} \\ Kz_{2j} \\ K^2\boldsymbol{E}_j^T \end{pmatrix}.$$

Hence, it follows that the vector $t' \in \langle KT_0, KT_{k+1}\rangle$, meets the equality $\boldsymbol{M}'_j\boldsymbol{c}'_j = \boldsymbol{Z}'_j$. The equation for $t' \in \langle KT_0, KT_{k+1}\rangle$, of the quadratic regression spline with this vector is the cut-off polynomial:

$$x_j = G'_j(t') = c_j^{(1)} + \frac{1}{K}c_j^{(2)}t' + \frac{1}{K^2}c_j^{(3)}(t')^2 + \frac{1}{K^2}\sum_{r=1}^{k} c_j^{(r+3)}(t' - KT_r)_+^2. \tag{12}$$

The equation (12) for $t' \in \langle KT_0, KT_{k+1}\rangle$, represents the same regression curve in $\mathbb{R}^m = (0; x_1, x_2, ..., x_m)$ as the equation with the untransformed parameter:

$$x_j = G_j(t) = c_j^{(1)} + c_j^{(2)}t + c_j^{(3)}t^2 + \sum_{r=1}^{k} c_j^{(r+3)}(t - T_r)_+^2\,, \qquad t' \in \langle T_0, T_{k+1}\rangle.$$

We obtain analogous results for the case when $Q = 1$ or $Q = 3$, as well.

4 SOLUTION TO PARTICULAR PROBLEMS

Example 4.1

We are supposed to solve a problem in which we reflect the aforementioned procedures. Our results were obtained by the computer program TRIO, created by the author of this article for the purposes of segmented regression, without which particular computations would be unfeasible by hand.

Table 1 Meteorological data

Time [h]		Temperature	Pressure	Wind
Real	Fictive	[°C]	[hPa]	[m/s]
6	1	15	800	5
8	2	16	850	4
10	3	17	900	3
12	4	22	1 000	2
14	5	28	1 030	1
16	6	26	1 020	2
18	7	20	950	3
20	8	19	900	3
22	9	18	890	3
24	10	16	840	4
2	11	15	820	4
4	12	13	810	5

Source: M. Kaňka

We assume that over time period from 6 am the values of the following three indicators: air temperature, air pressure and wind speed. The result of this measurement can be seen in Table 1.

In $\mathbb{R}^3$ (hence, $j = 1, 2, 3$) we have 12 experimentally obtained points, split in four groups (hence, $k = 3$) by the three points (we may call them morning, noon, evening and night group):

$$\begin{array}{lll} x_j^{(11)} = (15, 800, 5), & x_j^{(12)} = (16, 850, 4), & x_j^{(13)} = (17, 900, 3), \\ x_j^{(21)} = (22, 1000, 2), & x_j^{(22)} = (28, 1030, 1), & x_j^{(23)} = (26, 1020, 2), \\ x_j^{(31)} = (20, 950, 3), & x_j^{(32)} = (19, 900, 3), & x_j^{(33)} = (18, 890, 3), \\ x_j^{(41)} = (16, 840, 4), & x_j^{(42)} = (15, 820, 4), & x_j^{(43)} = (13, 810, 5), \end{array} \tag{13}$$

to which we assign the following values of a fictitious time:

$$\begin{array}{llll} t_{11} = 1, & t_{12} = 2, & t_{13} = 3, & \text{thus } n(1) = 3, \\ t_{21} = 4, & t_{22} = 5, & t_{23} = 6, & \text{thus } n(2) = 3, \\ t_{31} = 7, & t_{32} = 8, & t_{33} = 9, & \text{thus } n(3) = 3, \\ t_{41} = 10, & t_{42} = 11, & t_{42} = 12, & \text{thus } n(4) = 3, \end{array} \tag{14}$$

it holds that $\sum_{l=1}^{k+1} n(l) = \sum_{l=1}^{4} n(l) = 3 + 3 + 3 + 3 = 12,$ which is the total number of considered points the values of which may be $T_1 = 4$, $T_2 = 7$, $T_3 = 10$, together, for example, with additional time moments $T_0 = 1$ and $T_4 = 13$.

For example, for $Q = 1$, the matix $\boldsymbol{M}$ of system (8) is a $(k + Q + 1) \times (k + Q + 1) = 5 \times 5$ matrix. To save space, we neither give its full expression, nor for the 5-dimensional vectors $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_3$ on the right-hand side of this equation. This computationally intensive work was conducted by the computer program TRIO, that the author of this article created for the purposes of segmented regression on the basis of B-splines.

Nevertheless, for demonstration purposes, let us compute the element m_{43} of $\boldsymbol{M}$ and the element z_{53} of $\boldsymbol{Z}_3$. As $Q + 1 = 2 < 3 = q \leq k + Q + 1 = 5$, there will be, according to (6):

$$m_{43} = \sum_{l=1}^{4}\sum_{w=1}^{3}(t_{lw} - T_1)_+ \cdot (t_{lw} - T_2)_+ =$$

$$= \sum_{w=1}^{3}(t_{1w} - 4)_+ \cdot (t_{1w} - 7)_+ + \sum_{w=1}^{3}(t_{2w} - 4)_+ \cdot (t_{2w} - 7)_+ +$$

$$+ \sum_{w=1}^{3}(t_{3w} - 4)_+ \cdot (t_{3w} - 7)_+ + \sum_{w=1}^{3}(t_{4w} - 4)_+ \cdot (t_{4w} - 7)_+ =$$

$$= 0 + 0 + (7-4)_+ \cdot (7-7)_+ + (8-4)_+ \cdot (8-7)_+ + (9-4)_+ \cdot (9-7)_+ +$$

$$+(10-4)_+ \cdot (10-7)_+ + (11-4)_+ \cdot (11-7)_+ + (12-4)_+ \cdot (12-7)_+ =$$

$$= 0 + 0 + 0 + 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 + 7 \cdot 4 + 8 \cdot 5 = 100 = m_{34},$$

as $\boldsymbol{M}$ is symmetric, and further:

$$z_{53} = \sum_{l=1}^{4}\sum_{w=1}^{3} x_3^{(lw)}(t_{lw} - T_3)_+ =$$

$$= \sum_{w=1}^{3} x_3^{(1w)}(t_{1w} - 10)_+ + \sum_{w=1}^{3} x_3^{(2w)}(t_{2w} - 10)_+ +$$

$$+ \sum_{w=1}^{3} x_3^{(3w)}(t_{3w} - 10)_+ + \sum_{w=1}^{3} x_3^{(4w)}(t_{4w} - 10)_+ =$$

$$= 0 + 0 + 0 + x_3^{(41)}(t_{41} - 10)_+ + x_3^{(42)}(t_{42} - 10)_+ + x_3^{(43)}(t_{43} - 10)_+ =$$

$$= 0 + 0 + 0 + 4 \cdot (10-10)_+ + 4 \cdot (11-10)_+ + 5 \cdot (12-10)_+ =$$

$$= 0 + 0 + 0 + 0 + 4 \cdot 1 + 5 \cdot 2 = 14.$$

The computer program TRIO provides the parametric equations of the resulting (for our case $Q = 1$, linear) regression curve that we do not present here to save space.

The computation of the so-called determination indices, which is also provided by the program TRIO, yields for the aforementioned experiment (where Q = 1) the values $I_{x_1}^2 = 0.7539, I_{x_2}^2 = 0.9296, I_{x_3}^2 = 0.9009$.

This means that approximately 75% variability ofthe observed values x_1, 93% variability of x_2 and 90% variability of x_3 can be explained bya linear regression model. For $Q = 2$ and $Q = 3$ the program TRIO gives the following determination indices, see Table 2.

Table 2 Determination indices describing variability of observed values

Regression	$I_{x_1}^2$	$I_{x_2}^2$	$I_{x_3}^2$
Q=1	0.7539	0.9296	0.9009
Q=2	0.9302	0.9748	0.9179
Q=3	0.8998	0.9774	0.9474

Source: M. Kaňka

With respect to the determination indices, one can consider a certain "optimal regression curve" the equation of which for x_1 is based on quadratic regression ($Q = 2$), the equation for x_2 and x_3 on cubic regression ($Q = 3$):

$$x_1 = G_1(t) = \begin{cases} 16.7449 - 2.6212t + 0.9998t^2 & \text{for } 1 \leq t < 4, \\ -27.0951 + 19.2988t - 1.7402t^2 & \text{for } 4 \leq t < 7, \\ 104.7835 - 18.3808t + 0.9512t^2 & \text{for } 7 \leq t < 10, \\ -103.7065 + 23.3172t - 1.1337t^2 & \text{for } 10 \leq t \leq 13, \end{cases}$$

$$x_2 = G_2(t) = \begin{cases} 897.9027 - 176.7071t + 93.6242t^2 - 10.7554t^3 & \text{for } 1 \leq t < 4, \\ -51.2941 + 535.1905t - 84.3502t^2 + 4.0758t^3 & \text{for } 4 \leq t < 7, \\ 1200.3472 - 1.2272t - 7.7191t^2 + 0.4267t^3 & \text{for } 7 \leq t < 10, \\ 1380.2472 - 55.1972t - 2.3221t^2 + 0.2468t^3 & \text{for } 10 \leq t \leq 13, \end{cases}$$

$$x_3 = G_3(t) = \begin{cases} 4.0879 + 1.9577t - 1.2539t^2 + 0.1537t^3 & \text{for } 1 \leq t < 4, \\ 19.3391 - 9.4807t + 1.6057t^2 - 0.0846t^3 & \text{for } 4 \leq t < 7, \\ -20.2431 + 7.4831t - 0.8177t^2 + 0.0308t^3 & \text{for } 7 \leq t < 10, \\ -9.7491 + 4.3331t - 0.5027t^2 + 0.0203t^3 & \text{for } 10 \leq t \leq 13. \end{cases}$$

For example, for $t = 8$ we get the point (18.6139, 914.9777, 3.0585) on the optimal regression curve, which lies "near" the point (19, 900, 3) to which the value of the parameter belongs. Or for $t = 4.5$ we obtain the point (24.5105, 1020.3789, 1.4822) on the optimal regression curve. We may draw the conclusion that at 1 pm local time, the air temperature was approximately 25°C, the air pressure approximately 1020 hPa and the speed of the wind approximately 1 m/s.

Example 4.2

We shall solve the problem from Example 4.1 with the help of the polygonal method together with a transformation of the parametric variable (see Sections 2, 3). We assign to the points (13), which were arranged into four groups ($k = 3$), values $\tilde{t}$:

$$\tilde{t}_{11} = 0.00, \quad \tilde{t}_{12} = 50.02, \quad \tilde{t}_{13} = 100.04, \quad \text{thus } n(1) = 3,$$

$$\tilde{t}_{21} = 200.17, \quad \tilde{t}_{22} = 230.78, \quad \tilde{t}_{23} = 241.03, \quad \text{thus } n(2) = 3,$$

$$\tilde{t}_{31} = 311.29, \quad \tilde{t}_{32} = 361.30, \quad \tilde{t}_{33} = 371.35, \quad \text{thus } n(3) = 3,$$

$$\tilde{t}_{41} = 421.40, \quad \tilde{t}_{42} = 441.42, \quad \tilde{t}_{43} = 451.67, \quad \text{thus } n(4) = 3.$$

We easily find out that the number $\tilde{t}_{lw}$ expresses the length of the polygonal trail measured from the initial point $x_j^{(11)}$ to the considered point $x_j^{(lw)}$ ($l = 1,2,3,4$; w = 1,2,3), see (13). According to (10), in this case there is:

$$P = \min_{l=1,2,3}\{\tilde{t}_{l+1,1} - \lfloor \tilde{t}_{l,n(l)} \rfloor\} = \min\{100.17, 70.29, 50.4\},$$

thus p = ⌊P⌋ = 50, hence the knots of the problem are $\tilde{T}_0 = 0, \tilde{T}_1 = \lfloor 100.04 \rfloor + 50 = 150$, $\tilde{T}_2 = \lfloor 241.03 \rfloor + 50 = 291$, $\tilde{T}_3 = \lfloor 371.35 \rfloor + 50 = 421$, $\tilde{T}_4 = \lfloor 451.67 \rfloor + 50 = 501$. Through the transformation $t' = K\tilde{t}$, where $K = 1/(\tilde{T}_{k+1} - \tilde{T}_0) = 1/(\tilde{T}_4 - \tilde{T}_0) = 1/501$, the latter values of $\tilde{t}_{lw}$ are mapped onto $t'_{lw}, = K\tilde{t}_{lw} = \tilde{t}_{lw}/501$ and the knots $\tilde{T}_0 = 0, \tilde{T}_1 = 150$, $\tilde{T}_2 = 291$, $\tilde{T}_3 = 421$, $\tilde{T}_4 = 501$ onto $T'_0 = 0, T'_1 = 0.3$, $T'_2 = 0.58, T'_3 = 0.84$, $T_4 = 1$. Then, a subsequent execution of regression for Q = 1,2,3 provides (through the computer program TRIO) the following table of determination indices:

Table 3 Determination indices describing variability of observed values

Regression	$I^2_{x_1}$	$I^2_{x_2}$	$I^2_{x_3}$
Q=1	0.8017	0.9584	0.9009
Q=2	0.9193	0.9953	0.9308
Q=3	0.9004	0.99	0.9307

Source: M. Kaňka

With respect to the measures of the determination indices, it can be seen that the equations of the optimal regression curve for x_1,x_2,x_3 are according to the quadratic regression ($Q = 2$):

$$\begin{aligned} x_1 &= G_1'(t') \\ &= \begin{cases} 15.3435 - 12.2933t' + 107.2952(t')^2 & \text{for } 0 \le t' < 0.3, \\ -9.0168 + 150.4338t' - 164.4590(t')^2 & \text{for } 0.3 \le t' < 0.58, \\ 66.5740 - 109.8479t' + 59.5979(t')^2 & \text{for } 0.58 \le t' < 0.84, \\ -476.4334 + 1182.5355t' - 709.3856(t')^2 & \text{for } 0.84 \le t' \le 1, \end{cases} \\ x_2 &= G_2'(t') \\ &= \begin{cases} 801.6375 + 393.7033t' + 516.9006(t')^2 & \text{for } 0 \le t' < 0.3, \\ 546.2165 + 2099.9158t' - 2332.4744(t')^2 & \text{for } 0.3 \le t' < 0.58, \\ 1431.8960 - 949.7437t' + 292.7480(t')^2 & \text{for } 0.58 \le t' < 0.84, \\ 700.7491 + 790.4207t' - 742.6705(t')^2 & \text{for } 0.84 \le t' \le 1, \end{cases} \\ x_3 &= G'_3(t') \\ &= \begin{cases} 5.0080 - 10.8967t' + 5.2201(t')^2 & \text{for } 0 \le t' < 0.3, \\ 7.0535 - 24.5605t' + 28.0387(t')^2 & \text{for } 0.3 \le t' < 0.58, \\ -5.6486 + 19.1765t' - 9.6112(t')^2 & \text{for } 0.58 \le t' < 0.84, \\ 193.8010 - 455.5231t' + 272.8407(t')^2 & \text{for } 0.84 \le t' \le 1. \end{cases} \end{aligned} \tag{15}$$

E.g., for $t'_{32} = \tilde{t}_{32}/501 = 361.30/501 = 0.7212$ we get (18.3503, 899.2077, 3.1824) on the optimal regression curve that lies “near” the point $(19,900,3) = t_{32}$, see (13). Or for $t'_{43} = \tilde{t}_{43}/501 = 451.67/501 = 0.9015$ we obtain the point (13.1031, 809.7434, 4.8852) that lies “near” the point $(13,810,5) = t_{43}$, see (13).

We might also be interested in the question how to determine for $t = 4.5$, which lies between $t_{21} = 4$ and $t_{22} = 5$, see (14), the value t' lying between $t'_{21}/501 = 200.17/501 = 0.3995$ and $t'_{22}/501 = 230.78/501 = 0.4606$. Through the function:

$$\tilde{t} = \tilde{t}_{22} + \frac{\tilde{t}_{22} - \tilde{t}_{21}}{5 - 4}(t - 5),$$

that maps the interval $\langle 4,5 \rangle$ for the variable t onto $\langle \tilde{t}_{21}, \tilde{t}_{22} \rangle$ for $\tilde{t}$ we obtain for $t = 4.5$ the value:

$$\tilde{t} = 230.78 + \frac{230.78 - 200.17}{1}(-0.5) = 215.4750.$$

The transformation of the interval $\langle \tilde{T}_0 = 0, \tilde{T}_4 = 1 \rangle$ onto $\langle 0,1 \rangle$ is then done through the function:

$$t' = 1 + \frac{1 - 0}{501 - 0}(\tilde{t} - 501),$$

which yields then after the substitution $\tilde{t} = 215.4750$ that:

$$t' = 1 + \frac{1}{501}(215.4750 - 501) = 0.4306.$$

Inserting this value into (15), we obtain on the optimal regression curve that is, in comparison with the data in Table 1, quite acceptable.

CONCLUSION

Author was paying close attention to so called polygonal method of assigning values of parametric variable to experimentally obtained points, which displays those points with an oriented graph, and as a parameter of considered point it selects the length of polygonal trail measured from the initial point to the considered point. The main result of this procedure involves automatic computation of nodal points, which are associated with given task.

In order to improve numerical stability of equations of regressive curve, which is associated with the solution of specific task, it is recommended in literature to transform parametric interval to an interval of length one.

At the end of the paper the author introduces term optimal regression, which takes into account values of indexes of determination of observed units in specific model ($Q = 1,2,3$).

Individual stages of solution concerning the task of segmented regression are demonstrated in an example from meteorology. Introduced computations are product of computer program TRIO, without which the computations (starting with the elements of matrix of system of normal equations) would be hardly feasible.

References

BÉZIER, P. *Numerical Control: Mathematics and Applications.* London: Wiley, 1972.

BÖHMER, J. *Spline-Funktionen.* Stuttgart: Teubner, 1974.

FEDER, P. I. On Asymptotic Distribution Theory in Segmented Regression Problems – Identified Case. *The Annals of Statistics*, 3 (1975), pp. 49–83.

GUZIK, B. Estymatory horyzontu prognozy v trendach segmentiwych. *Przegled Statyczny*, 21 (1974).

KAŇKA, M. Segmented Regression Based on B-Splines with Solved Examples. *Statistika*: *Statistics and Economy Journal*, 95(4), 2015, pp. 39–58.

MAKAROV, V. L., CHLOBYSTOV, V. V. *Splajn-approximacija funkcij.* Moscow: Vyssaja škola, 1983.

MELOUN, M., MILITKÝ, J. *Statistické zpracování experimentálních dat.* Prague: PLUS, s.r.o., 1994.

SCHMITTER, D., DELGADO-GONZALO, R., UNSER, M. A family of smooth and interpolatory basis functions for parametric curve and surface representation. *Applied Mathematics and Computation*, New York, 2016, Vol. 272, pp. 53–63.

SEGER J. *Statistické metody pro ekonomy průmyslu.* Prague: SNTL / ALFA, 1988.

SUNG, M. *3D Motion Editing Through B-spline Fitting with Constraints.* Proceedings of the 3rd international conference on intelligent technologies and engineering systems (ICITES2016), Taiwan, 2016, Vol. 345, pp. 167–173.

SPÄT, H. *Spline Algorithmen zur Konstruktion glatter Kurven und Flächen.* 4th Ed. München–Wien: Oldenbourg, 1996.

VASILENKO, V. A. *Splajn-funkcii, těorija, algoritmy, programy.* Novosibirsk: Nauka, 1983.

WANG, X., WU, Z., SHEN, J., ZHANG, T., MOU, X., ZHOU, M. *Repairing the cerebral vascular through blending Ball B-spline curves with G(2) continuity.* Amsterdam: Neurocomputing, 2016, pp. 768–777.