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Impact of the Implementation of ESA 2010 on Volume Measurement

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Abstract

Volume indices are connected with statistical deflation that means recalculation of macro-aggregates to constant prices. Price calculations have to follow changes in definition or delineation of macro-aggregates. New standards of National Accounts (SNA 2008, ESA 2010 respectively) bring many changes that should be taken into account in volume measures. The aim of this paper is to present new methods of deflation that respect updated definitions and principles. Concept of foreign trade has been changed significantly as globalization is going faster and faster. Re-export and merchanting have become more important especially in small open economies such as the Czech Republic. This phenomenon should be reflected in constant prices calculations. Changes in methodology have also affected volume indices.

Keywords	***JEL code***
National accounts, ESA 2010, revision, research and development, foreign trade	*E01*

INTRODUCTION

System of National Accounts is macroeconomic statistical model that is designed for the description of economy. National accounts provide data on production, generation of income, its distribution and redistribution as well as accumulation. History of National Accounts started in the 18th century, when François Quesnay published the Economic Table (Tableau économique). National Accounts[3] have been improving since and the first international framework (SNA 1952) was published in 1952 (Hronová et al., 2009). Standards have to be updated regularly as the economy has been changing quickly, especially in recent years. The latest international standard is SNA 2008. European standard ESA 2010, which is derived from SNA 2008, became effective in September 2014. New standards SNA 2008, ESA 2010, respectively, brings significant changes of concept of productive activity and definition of assets

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3 Balances of National Income (BNI) were used instead of National Accounts in communist regimes including Czechoslovakia. Definition of productive activity is narrower in BNI than in National Accounts. Transformation of data from BNI to National Accounts was carried out by researchers from University of Economic in Prague (Sixta et al., 2014).

(Sixta, 2014). There were many discussions on impacts of revisions on value of macro-aggregates, structure of input-output tables, balance of payment etc. Unfortunately, almost no attention was paid to the impact on volume measures though volume indices are under spot light. Countries have to prepare new deflation procedures independently as there were no international recommendations on volume estimates of newly defined items.

Statistical deflation means transformation of indicators at current prices to constant prices in order to eliminate price changes and estimate volume indices. However, number of indicators that can be revaluated to constant prices is limited as no suitable price indices are defined for many indicators (Nicolardi, 2013). Therefore the paper is focused on impact of changes on transactions in products.

1 PARTIAL CHANGES

The list of changes between current standards of National Accounts and previous standards is long. However, many changes are insignificant and they can be considered as clarification of terminology with negligible impact on indicators. Some changes may be important in a few countries. Nevertheless, selected changes are important in all countries, such as capitalization of military expenditure, small tools, Research and Development, new concept of foreign trade and new concept of output of insurance services.

1.1 Research and Development

Research and Development (R&D) is now identified as a fixed asset in National Accounts. It means that acquisition and disposal of R&D is recorded as gross fixed capital formation. Moreover, Research and Development, as any other fixed assets, is depreciated. R&D is defined in the 'Manual on measuring Research and Development in ESA 2010'[4] as follows: 'Research and Development is a creative work undertaken on systematic basis to increase the stock of knowledge, and use of this stock of knowledge for the purpose of discovering or developing new products, including improved versions or qualities of existing products, or discovering or developing new or more efficient processes of production'.

Deflation techniques should reflect valuation of macro aggregates at current prices. The output of R&D at current prices is measured as follows (EUROSTAT, 2013):

a) R&D by specialized commercial research laboratories or institutes is valued at the revenue from sales, contracts, commissions, fees, etc. in the usual way;
b) The output of R&D for use within the same enterprise is valued on the basis of the estimated basic prices that would be paid if the research was subcontracted. In the absence of a market for subcontracting R&D of a similar nature, it is valued as the sum of production costs plus a mark-up (except for non-market producers) for net operating surplus (NOS) or mixed income;
c) R&D by government units, universities and non-profit research institutes is valued as the sum of costs of production. Revenues from the sale of R&D by non-market producers are to be recorded as revenues from secondary market input.

Expenditure on R&D is distinguished from expenditure on education and training. It does not include the costs of developing software as the main or secondary activity. Basic rule of R&D determination is the presence of novelty, creativity, orderliness, uncertainty and reproducibility in R&D. According to the Frascati Manual (OECD, 2002) the previously mentioned activities are not included.

The most important part of output of Research and Development is own account produced R&D in the Czech Republic. Valuation of this output is similar to the approach to output of other non-market services. Therefore, deflation technique can be analogical. Other non-market output is deflated using the input method. This approach is based on deflation of each component separately (EUROSTAT, 2001).

[4] EUROSTAT (2014a, p. 6).

However, input method does not enable to carry out analysis of productivity and effectiveness. It means that change in output is equal to changes in inputs (Atkinson, 2005). On the other hand, definition of price representatives for R&D is almost impossible and Research and Development is considered to be collective product for which input method can be used though it is not an ideal solution (EUROSTAT, 2001, p. 113). Market output and output for own final use is deflated by appropriate price indices (PPIs, price indices of export, etc.).

Countries apply different approaches to deflation of capitalised Research and Development. Ritter (2014) has published the paper focused on the approach used in Germany.

1.1.1 Price and Volume Measurement for R&D in German National Accounts[5]

About ¾ of the total R&D output in Germany is produced by nonfinancial corporations. The R&D output of financial corporations is really unimportant (less than 1%). About 20% of the total R&D output is produced by the general government and about 4% by the non-profit institutions serving households.

Input method has been already used for deflation in the government and non-profit institutions sector. The approach is applied on Research and Development in these sectors. However, this method has not been used in other sectors yet. In German national accounts R&D output of nonfinancial and financial corporations is calculated for industries in the following way based on data of the Stifterverband:[6]

Table 1 Estimate of output of Research and Development

	Estimate of R&D output
	Intramural expenditure on R&D
-	Capital expenditure on R&D
=	Current expenditure on R&D
+	Other taxes on production
-	Other subsidies on production
+	Consumption of fixed capital
+	Operating surplus, net
=	R&D output including R&D for software
-	R&D for software
=	R&D output without purchase of R&D for intermediate consumption in the industry A72 (main production R&D)
+	Purchase of R&D from non-financial and financial corporations for intermediate consumption in the industry 72 (NACE Rev. 2)
=	R&D output

Source: Ritter (2014)

5 This chapter is based on Ritter (2014).

6 The Stifterverband für die Deutsche Wissenschaft (Association of funders for the German science) is a private non-profit institution. Yearly reports about its R&D survey are published by the Wissenschaftsstatistik GmbH.

The price and volume measurement for intermediate consumption of nonfinancial and financial corporations are based on use tables. Yearly deflators for total intermediate consumption are calculated in a breakdown by industries and product groups. These deflators can be used for deflating intermediate consumption for R&D output as well.

The German use tables distinguish between 64 industries and 88 product groups. For the industry NACE 72 Scientific research and development the input structure of total intermediate consumption – subdivided by product groups – is representative for the input structure of intermediate consumption for R&D output. Only for this industry the input structure can be taken from the published use tables without any modification. Principles to estimate the input structure for other industries are as follows: The structure of intermediate consumption for R&D output can be derived from the structure of total intermediate consumption of the industry which generates the R&D output. The structure of intermediate consumption of the whole industry cannot be applied to R&D output without modifying it. In doing so data about the cost structure of the industry 72 'Scientific research and development' (NACE Rev. 2) can serve as reference figures.

The input method for compensation of employees is based on deflators derived from weighted average for gross hourly earnings of R&D staff by levels of qualification. There are three staff categories defined by the Frascati Manual:[7] Researchers, Technicians and equivalent staff, other support staff. The quarterly earnings survey (performance groups of employees with a similar job qualification profile) identifies these five so called "Performance groups" (PG): PG 1 Managing directors, PG 2 Senior skilled workers, PG 3 Specialised personnel, specialists, skilled staff, PG 4 Semi-skilled workers, PG 5 Unskilled workers.

1.1.2 Price and Volume Measurement for R&D in Czech National Accounts

The main data source on Research and Development is statistical survey VTR 5-01, which is based on Frascati Manual. Besides, special questions on subcontracts are included for National Accounts calculations. Estimates at current prices are in line with Eurostat Manual on measuring Research and Development in ESA 2010. Special deflation techniques had to be developed as the concept has changed. Non-market output of R&D is not so important (about CZK 10 mil. in 2013) and it is a part of final government expenditure (not GFCF). Market output is deflated by index of average compensation of employees. There is no price index (e.g. PPI) because it is almost impossible to define a price representative. Therefore this method is considered to be suitable (EUROSTAT, 2001, p. 105) and it was used in the past as well (CZSO, 2008). However, new deflation method for own account produced R&D has to be developed. Czech approach is similar to (and inspired by) German method, which means that each type component is deflated separately. Compensation of employees is deflated by index of average compensation of employees in related industry. Currently, we do not do any stratification by staff categories, but we are investigating whether data in both dimensions (staff category, industry) is possible to gather. Intermediate consumption (IC) is recalculated by implicit deflator of IC in NACE 72. As supply and use tables are compiled for each version of annual National Accounts, possible changes in structure IC is included in the deflator. Another option would be to use symmetric input-output tables SIOT) because the main part of output of R&D is produced in other industries (e.g. production of industrial products, education services). Symmetric input-output tables offer product structure of costs related R&D product (not industry). Nevertheless, data is in basic prices and has to be transformed to basic prices and SIOT is not compiled annually. Consumption of fixed capital is estimated directly at constant prices within PIM.[8] Currently, other components are deflated by implicit deflator of output. We plan to improve this approach and use price index for market R&D.

7 Ritter (2014).

8 Perpetual Inventory Method. Consumption of fixed capital is estimated using actual service life of assets, for detail see Sixta (2007).

1.2 Small tools

Fixed assets used to be defined in ESA 1995 as items used in production for more than one year with value higher than 500 ECU. Purchase of items below this threshold was classified as intermediate consumption. Now, in ESA 2010, no such threshold is given, the only criterion is the use in production process for more than one year. The change in value added is opposite to the change in intermediate consumption (production approach) and equal to the change in gross fixed capital formation (expenditure approach) and in gross operating surplus (income approach). This change has no impact on deflation techniques, as intermediate consumption and gross fixed capital formation are deflated in the same way. On the other hand, GDP was affected significantly, about 1.9% in 2013.

1.3 Foreign Trade

Concept of foreign trade has been altered significantly though the main principle (a change in ownership) remains unchanged. The main reason is that globalization is going on quickly and economic statistics has to follow it.

1.3.1 Processing

Between ESA 1995 and ESA 2010, there is a fundamental change in the treatment of goods sent abroad for processing. According to the previous standard (ESA 1995), such goods were shown as exports because of the fact they were sent abroad, and then recorded as imports on return from abroad, at a higher value as a result of the processing. This was known as the gross recording method where international merchandise trade figures represent an estimate of the value of the goods being traded.

According to the ESA 2010 a change of ownership is not imputed but the processing service is recognized. Processing service is a part of export of services for the country where the processing takes place. This recording is more consistent with business accounting and associated financial transactions. However, it does cause an inconsistency with the international merchandise trade statistics (IMTS). This statistics will continue to record gross value of exports for processing and returning imported processed goods, as it is based on the physical movement of goods, rather than the change of the economic ownership of the goods.

This change results in one main consequence: a new processing service is recognized. It means that value of goods sent for (after) processing is not included in export and import. Deflation methods have to follow changes in current prices. In the past, inward processing (export and import) was deflated by price indices of import. The same type of price indices was used on both sides (export, import) in order not to influence a balance of foreign trade and also gross value added. Similarly, outward processing was deflated by PPIs. Currently, just processing service is deflated. Used price indices remain the same, however, the interpretation is different. It can be argued that price development of services and goods may vary. On the other hand, no better method[9] has been introduced yet.

1.3.2 Merchanting

Standard ESA 2010 defines export of merchanting as follows: The purchase of a good by a resident from a non-resident and the subsequent resale of the good to another non-resident, without the good entering the merchant's economy. Import of merchanting is not defined explicitly, but as analogous case in import. Export of merchanting is recorded on 'net' principle, i.e. export of margin (sales minus cost

[9] Eurostat established task force on price and volume measures that started in 2015. This method will be accepted in updated Manual on Price and Volume Measures as suitable method. Other possible approaches are: deflation by index of wages or input method. However, cost structure of processing service is not available in most countries.

on sold products). Theoretically, it can be negative if costs are higher than sales. Unfortunately, import of merchanting, which is a mirror case in partner country, has not been covered by international manuals. The Czech Statistical Office has promoted that at many international meetings. It is called inverse or negative merchanting.

As merchanting is in fact trade margin it can be deflated similarly. However, it can be merchandised products that have normally no trade margin, e.g. electricity. Moreover, basic prices of merchanting of a particular product (e.g. electricity) are zero that does not enable to apply standard method of deflation of trade margin used by the Czech Statistical Office.[10] Several approaches were broadly discussed at the CZSO but also with colleagues from other countries. Finally, the Czech Statistical Office has introduced own method that has been promoted at international meetings.[11] It takes into account product structure of goods that are merchandised and also territorial structure of transactions. Sales and costs are deflated separately and export of merchanting is calculated as a difference between sales at previous year's prices and costs at previous year's prices. It is similar to the double deflation of gross value added. Sales are deflated by price indices of countries where specific products are sold. Price indices are also adjusted to changes in exchange rate of given countries. Similar approach is applied to deflation of costs. Inverse (negative) merchanting is deflated by price indices of import. 'Double deflation' method is not used as the structure of countries is not available directly from data sources. However, a model approach to preparation of the structure of countries is planned to be used in the future. It should enable to apply 'double deflation' method for inverse merchanting as well.

1.3.3 Re-export

Re-export is defined as goods, that are produced abroad and imported into the domestic territory by residents (so a change in ownership from non-resident to resident occurs) and that are subsequently without significant transformation exported abroad (so again a change of ownership from resident to non-resident occurs). The goods cross the border of domestic territory and are therefore recorded in the foreign trade statistics. Re-export is mentioned as a globalisation phenomenon in ESA 2010. According to change of ownership principle the re-export is considered as export and import of goods in the National Accounts (CZSO, 2014). Although the principle is similar to merchanting, the recording in National Accounts is different (gross principle). The difference between merchanting and re-export can be found in territory (domestic for re-export, foreign for merchanting) but the nature of the activity is the same. Value of re-export in export and import varies, the difference makes the trade margin. Value of good itself is deflated by price indices of export, margin is deflated separately. A rate of margin from the previous year is applied to good itself at previous year's prices. It is fully consistent with deflation of margin related to other types of uses.

1.4 Other changes

Standard ESA 2010 brings many other changes but they no or negligible impact on volume measures. Delineation of government sector may have an impact on method of calculation of output (market or non-market). Subsequently deflation method is changed. Weapon systems are recognized as an asset therefore government expenditure on them is considered as gross fixed capital formation. It has also an impact on other non-market output and government consumption that are deflated by input method. Other changes do not have an impact on transaction in products or do not require changing deflation methods.

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[10] For more details about deflation of trade margin see CZSO (2008).

[11] This method was presented at above mentioned task force. Now it is being discussed and it will be probably accepted as an appropriate deflation method (method A).

2 IMPACT OF CHANGES ON VALUE AND VOLUME INDICES

Changes in methodology and deflation techniques cause changes in value and volume indices. Implicit deflators were also affected as product structure of indicators changed. The impact of the most important changes was separated for gross value added and gross fixed capital formation that were mainly affected. Changes that occurred within the revision are quantified in GNI Questionnaire.[12] However, all changes had to be also estimated at previous year's prices. Developed deflation techniques were applied to merchanting, Research and Development and re-export. Other items were deflated by appropriate price index related to the item (e.g. small tools by deflators of gross fixed capital formation in product breakdown). Consumption of fixed capital related to the capitalisation of R&D was estimated at previous year's prices directly within PIM method. Contributions to growth method was employed, see the following formula:

$$c = \frac{I_q}{w}\ , \qquad (1)$$

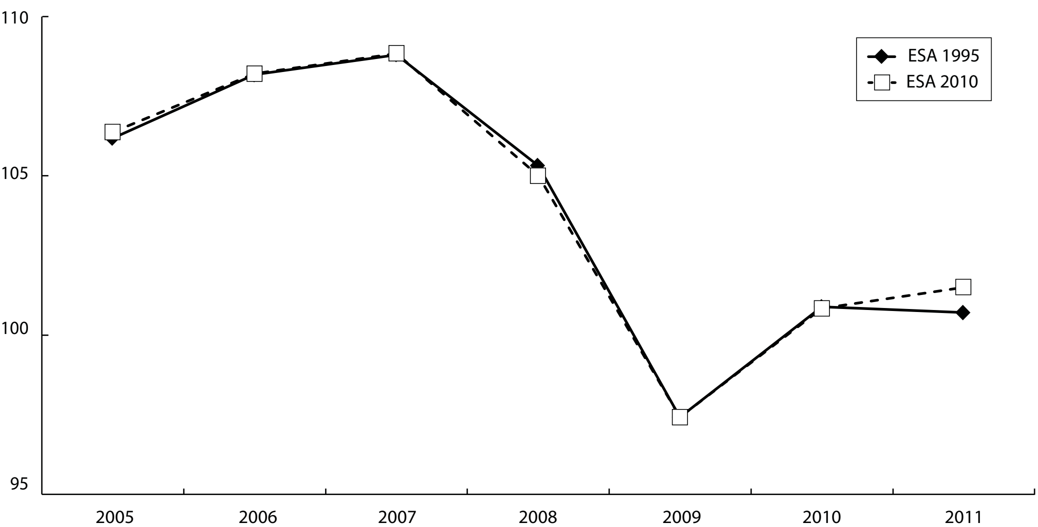
where c denotes the contribution of particular macro-aggregate, I_q denotes to volume index of particular macro-aggregate expressed in percentage and w denotes to a share of particular macro-aggregate on GDP in previous year.

However, a reconciliation of particular results with overall results had to be done. A share of ESA 2010 changes on GDP differs year by year (from 3.0% to 6.5%).[13] Shares of particular ESA 2010 changes (e.g. R&D) on total changes depends on a share of R&D on ESA 2010 changes but also on ESA 2010 changes on total changes.

2.1 Changes in indices of Gross Value Added

The differences in value indices of gross value added are negligible, see Figure 1.

Figure 1 Value indices of Gross Value Added



Source: CZSO; authors' computation

[12] Member states of EU are obliged to report GNI Questionnaire annually. Moreover, they have to quantify changes between ESA 2010 methodology and ESA 1995 methodology.

[13] See: <https://apl.czso.cz/nufile/Uvodni_poznamky_01_10_2014.pdf>.

The revision did not cause significant changes in volume indices of gross value added (GVA). The difference in volume index is less than 0.5 p.p. in all years. Research and Development and small tools increase volume index in most years, on the other hand changes in foreign trade have negative impact on volume index.

Table 2 Changes in volume indices of Gross Value Added

Gross Value Added	2005	2006	2007	2008	2009	2010	2011
ESA 2010	6.60	7.50	5.20	3.59	–5.49	2.87	1.97
ESA 1995	6.99	7.70	5.49	4.06	–5.16	3.12	1.81
Difference (p.p.)	–0.39	–0.20	–0.29	–0.47	–0.34	–0.25	0.16
Infl. of changes ESA 2010	0.11	–0.16	0.11	–0.48	-0.11	–0.03	0.06
R&D	0.06	0.00	0.07	0.00	0.07	–0.03	0.07
Small tools	0.23	0.42	0.12	–0.08	–0.05	0.04	0.24
Weapon systems	0.02	–0.03	–0.02	–0.02	–0.01	–0.01	0.00
Changes in foreign trade	–0.07	–0.04	–0.05	–0.12	0.04	–0.05	0.00
Insurance services	0.23	–0.35	0.08	–0.10	–0.02	0.19	–0.26
Other changes by ESA 2010	–0.34	–0.16	–0.10	–0.16	–0.14	–0.18	0.01
Infl. of other changes	–0.50	–0.04	–0.40	0.00	–0.22	–0.22	0.10
Improvement/Other	–0.54	–0.02	–0.41	0.08	–0.13	–0.43	0.07
Balancing adjustments	0.04	–0.02	0.01	–0.07	–0.09	0.21	0.04

Source: CZSO; authors' computation

2.2 Changes in indices of Gross Fixed Capital Formation

Table 2 shows impact of revision on value indices of gross fixed capital formation. The highest difference (1.45 p.p.) is observed in 2009. The decrease is now smaller than it was according to ESA 1995. The difference is caused mainly by Research and Development and Weapon systems which are acquired at least partly by government institutions. It is known that government investment is more stable than investments of companies. As a consequence a decline of gross fixed capital formation is less deep than it was.

Table 3 Changes in value indices of Gross Fixed Capital Formation

GFCF	2005	2006	2007	2008	2009	2010	2011
ESA 2010	7.07	6.63	15.20	2.91	–8.74	0.24	0.28
ESA 1995	5.97	6.91	15.05	4.20	–10.19	0.48	–0.85
Difference (p.p.)	1.11	–0.27	0.15	–1.29	1.45	–0.24	1.13
Infl. of changes ESA 2010	1.22	–0.26	0.06	–0.44	1.30	0.07	1.44
R&D	–0.09	–0.41	0.27	0.04	0.76	–0.07	0.40
Small tools	0.59	1.47	–0.07	–0.44	0.08	–0.03	0.93
Weapon system	0.73	–1.32	–0.14	–0.05	0.46	0.17	0.10
Infl. of other changes	–0.12	–0.01	0.09	–0.85	0.15	–0.32	–0.30
Improvement/Other	0.71	–0.05	–0.58	–0.89	0.20	–0.08	–2.07
Balancing adjustments	–0.82	0.04	0.68	0.04	–0.05	–0.23	1.77

Source: CZSO; authors' computation

Changes in volume indices may differ from changes in value indices as new deflation techniques have been introduced. Research and Development contributed to the growth of GFCF at current prices by 0.27 p.p. in 2009. However, the effect in volume index is negative as the increase in current prices was caused by a change in price level. Contributions of capitalisation of small tools are positive in all monitored years with exception of 2008. It is probably caused by changes in production process that is being modernized and requires more ICT.

Table 4 Changes in volume indices of Gross Fixed Capital Formation

GFCF	2005	2006	2007	2008	2009	2010	2011
ESA 2010	6.41	5.87	13.54	2.54	–10.09	1.32	1.07
ESA 1995	6.03	5.80	13.24	4.10	–11.05	1.02	0.36
Difference (p.p.)	0.38	0.07	0.30	–1.57	0.96	0.30	0.72
Infl. of changes ESA 2010	1.39	0.03	0.11	–0.05	1.12	–0.09	1.18
R&D	–0.08	–0.25	–0.02	0.02	0.84	–0.27	0.20
Small tools	0.72	1.45	0.18	–0.04	0.02	0.02	0.91
Weapon system	0.74	–1.17	–0.05	–0.03	0.27	0.15	0.07
Infl. of other changes	–1.01	0.05	0.19	–1.52	–0.16	0.39	–0.46
Improvement/Other	0.18	–0.02	–0.52	–0.98	0.24	0.27	–2.00
Balancing adjustments	–1.19	0.06	0.71	–0.54	–0.41	0.12	1.54

Source: CZSO; authors' computation

CONCLUSION

The paper is focused on changes that have been brought by new standard ESA 2010 and subsequent changes in deflation techniques. Although new standard should have been fully implemented by September 2014, international discussion on price and volume measurements started in the following years. It was obvious that deflation techniques should be changed in order to follow the new concept of indicators. Main changes are described in this paper as well as newly developed deflation techniques. They have been implemented the time series (1990 onwards). Nevertheless, some simplifications had to be done for the beginning of the time series due to insignificance of changes or lack of data.

The impact of changes was estimated. It was a difficult task as some changes have also indirect impact. Research and Development, small tools or weapon system cause changes in non-market output via consumption of fixed capital. The impact of some changes in volume index of GVA (e.g. capitalization of small tools) is cyclic. It is negative in the years of crisis (2008 and 2009) and positive in other years. The impact of capitalization of R&D is almost always positive because it considered crucial factor for the economy and it is supported by various economical tools (subsidies, taxation etc.) However, the impact of some items is accidental and depends on factors outside the economy, e.g. changes in insurance services are brought by natural disasters. We can conclude that the development of economy is similar but not the same.

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Pilot Application of the Dynamic Input-Output Model. Case Study of the Czech Republic 2005–2013

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Abstract

The aim of this article is to provide the very first analysis in the field of dynamic Input-Output (I-O) models for the Czech Republic. This study examines the practicality of production dynamic equations for an estimation of future production enhanced for gross fixed capital formation. The principal construction element of dynamic I-O models rests on a technical capital matrix illustrating a stock of gross fixed capital in an economy. The lack of available data for this matrix challenges this study to analyze two possible computation procedures. Namely, I examine extrapolation method and method based on a transformation from matrix classification by type of fixed assets (AN) to classification by product (CPA). The results of the application part indicate notable differences between both ways of calculation. Final prognosis of the structure of production exhibits 11 to 21% deviations from the real structure of production in the five-year period and thus significantly diverges from reality. Potential sources of these problems and their solutions are discussed in the conclusion of this study.

Keywords	***JEL code***
Dynamic Input-Output model, Input-Output analysis, matrix of technical capital, production equation	*C67, D24, D57*

INTRODUCTION

Although, one might nowadays regard the elementary static Input-Output models as an outdated concept for modelling structural relationships in an economy, advanced macroeconomic models commonly base their assumptions on the equations and relations stemming from these I-O models. Examples of such macroeconomic models comprise models constructed on the basis of dynamic equations such

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as INFORUM (Inforum, 2015), INFORGE (Lutz, Ch. et al., 2003), or DSGE models enhanced for Input-Output data (Bouakez et al., 2005, 2009) or for elementary relationship expressed by Leontief matrix. A relatively wide range of models then seems to utilize especially production side of Input-Output models.

Dynamic Input-Output models enlarge basic static ones and diverge from static approach in time. Next, they contain additional information, namely, the information about technical capital. This capital constitutes the growth part of the model, which critically establishes the direction of evolution for total production.

While, the intermediate consumption matrix was in the center of attention for static I-O analysis, the dynamic version partially diverts its scrutiny to the technical capital matrix **B**. Most of the research papers (Inforum, 2015) examining I-O models devote a substantial part of their studies to this matrix for several reasons. First motivation is the problem of interpretation and uncertainty about the appropriate content of this matrix (Diaz, Carvajal, 2002). Second reason origins in commonly not favorable structure of the matrix, irregularity (Miller, Blair, 2009). The third important inquiry is the depreciation time of the capital and its varying influence on individual industries. Some attempts to solve this latter problem comprise of studies such as Idenburg, Wilting (2000) or Inforum (2015). I find the last and one of the main obstacles in the method of matrix development. Most countries do not publish matrices in classification by product or industry.

The technical capital matrix **B** is from the construction viewpoint of dynamic (not static) model more important than the matrix of technical coefficients **A** for the subsequent arguments. Apart from the above mentioned reasons the technical capital matrix determines the stability of economic system along with the growth potential of production (Díaz, Carvajal, 2002).

The results of those several models will be compared with forecast of total production for a basic static model and reality. In order to control for several factors originating in final consumption, I will substitute estimates of the model for real measured data. The resulting ex-post calculation will reveal prediction capability of dynamic models in context of static ones for total production in relation to final consumption.

The first chapter sums up the current state of knowledge regarding Input-Output dynamic methods. This part precedes a chapter summarizing methodological and theoretical characteristics of dynamic models – static foundations, dynamic analysis and derivation of technical capital matrices. Next, I concentrate on the problem of data sources. The results of application of these models on the data of the Czech Republic follow. The context of my results is discussed in the end of this paper.

1 LITERATURE REVIEW

Static Input-Output models find an implementation mostly in structural impact evaluation of interventions into economics with an emphasis on inter-industrial linkages. The contemporaneous Czech studies of such merit consist of VICERRO (2013) or the Ministry of the Environment of the Czech Republic (2014). The construction of static models resembles Keynesian ideas (Goga, 2009, pp. 26–32). Dynamic Input-Output models transmit the ideas further, allowing thus to incorporate long-run effects and trends across and between industries. The critics most commonly denounce the dynamic Input-Output models for an attempt to capture a dynamic non-static process (Lee, 2005) as snap shot of an economy at given time (Murray, 2011). Despite this rather negative evaluation, a broad area of application exists for a dynamic model in context of various research questions. For example consider Model DIMITRI (Idenburg, Wilting, 2000) scrutinizing the impact of interrelationship between economy, technology and the environment or environmental dynamic I-O models (Yokoyama, Kagawa, 2006; Dobos, Tallos, 2013).

The dynamic I-O models provide analysis ranging from topics such as inflation studies caused by national currency devaluation (Katsinos, Mariolis, 2012) to models combining Input-Output dynamic methods with so-called Grey system theory (Li, 2009). Other models completely forward the idea of structural analysis of I-O models into the context of DSGE models to develop detailed DSGE model based on dynamic I-O model (Bouakez et al., 2005, 2009) and capital matrix of I-O model.

One of the most important parts of the dynamic Input-Output models is the technical capital matrix. The article of Díaz, Carvajal (2002) provides an interesting study encapsulating diverse theoretical case studies in the context of dynamic Input-Output model and the technical capital matrix. The authors in their paper examine an effect of the technical capital matrix in specific situations such as in the case of zero willingness to invest or in the presence of lack of alternations of consumption or production in an economy. My study (Šafr, 2014) describes some of those cases in bigger detail.

Despite the long-lasting economic discussion regarding the technical capital in macroeconomic models in general (OECD, 2009), the effect of the technical capital matrix on the dynamic Input-Output model is an under-researched area (Pauliuk, Wood, Hertwich, 2015, p. 105; Leontief, 2007a, 2007b; Raa, 1986) summarize the basic understanding of the issue of capital inclusion into the production function of dynamic model and explain its context.

The matter of construction of states of the capital consists of four points. First task concerns the way of matrix construction. The second one questions the included variables. The third task pertains to the effects of the matrix on dynamic I-O model and the fourth problem covers singularity issue.

Mathematics enables an evasion of the fourth difficulty with the help of pseudo-inverse methods. Such methods can result in unstable outcomes as other authors indicate (Miller, Blair, 2009; or Šafr, 2014). Some authors solve this situation by an enlargement of specific methods or by refinement of contemporary pseudo-inverse methods (Jódar, Merello, 2010) or for example by succeeding calculation to obtain an invertible matrix (Sharp, Perkins, 1973). Subsequently, it is possible to solve this obstacle by combining dynamic I-O models with other approach (Zhang, 2000).

The second problem is even more complicated than the previous one concerning matrix singularity. National accounting quantifies the volume of gross fixed capital formation (GFCF) denoted as item P.51 in SIOT tables. The GFCF demonstrates the pure acquisition of fixed capital regardless its depreciation (Hronová et al., 2009). Although other studies favor additional inclusion of human capital next to the fixed capital into macroeconomic models (Zhang, 2008), in respect to the structural analysis in this study, our I-O model will not incorporate human capital. The following part examines the methodology for the I-O model outlined in this paper.

Regardless the latter problem, this study mainly inspects the first and the third point. These issues will be analyzed with help of variable for fixed capital.

2 METHODS A METHODOLOGY

2.1 Dynamic Input-Output model

2.1.1 Basic static I-O approach

SIOT tables usually represent the dataset for Input-Output models. Elementary static Input-Output model is based on linear relationships between production flow (intermediate consumption) from individual industry (i) to other industry (inputs-j) and between creation of production and production of a particular industry as whole. One can illustrate this link as (Goga, 2011, p. 75):

$$x_{i,j} = f_j(x_j), \qquad i, j = 1,2,3 \ldots\ldots n, \tag{1}$$

where $x_{i,j}$ stands for the flow of production from industry i to industry j. Variable x_j represents total production in the industry j. I assume a linearly definable relationship between x_j and $x_{i,j}$ and stable fixed ratio between $x_{i,j}$ and x_j in the long run.

Given this relationship one can define the elementary linkages of Input-Output models as:

$$\sum_{j=1}^{n} a_{ij}x_j + y_i = x_{i,} \text{ , matrix form: } \mathbf{Ax} + \mathbf{y} = \mathbf{x} \, . \tag{2}$$

Principal matrix representation of the model:

$$(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} = \mathbf{x}\,, \tag{3}$$

illustrates the link between final use (**y**) and total production (**x**) to satisfy the final use for the entire economy. Variable $a_{i,j}$ stands for the elements of the matrix **A** of a dimension n x n and illustrates technical coefficients of a production function. In other words, these elements symbolize the ratio between the input flow into a industry for creation of product and total production.

One can depict this relation in the following manner:

$$\text{Matrix: } \mathbf{A} = (a_{ij})_{n\times n}, \qquad \text{with elements: } a_{ij} = \frac{x_{ij}}{x_j}, \tag{4}$$

for which: $0 \leq a_{ij} \leq 1, \qquad i, j = 1,2, \ldots\ldots, n.$

Technical coefficients of the matrix **A** are assumed to be stable in the long run. This supposition appears in the production function form of this and subsequent period as:

$$f_j^t(x_j) \approx f_j^{t+p}(x_j). \tag{5}$$

Primary assumption about long-run stability of production function leads then to long-run stability of individual technical coefficients (a_{ij}).

This basic Input-Output model offers a wide range of potential applications. It is most often used for an analysis of structural linkages in an economy and impact evaluation of predominantly multiplication effects.

2.1.2 Dynamic I-O approach

Dynamic models with help of difference and differential equations extend the basic static Input-Output model for capital-flow matrix. This matrix aims to capture the influence of realized investments to technical capital on the growth of an economy. Next, its goal is to elude principal obstacles of static analysis (Goga, 2009, p. 103) considers a link between calculated parameters of one period in relation to exogenous parameters of the next one as the most crucial problem concerning the static model. The model thus neglects the impact of capital investments, such as purchases of new machineries or capacity expansions.

The core topic of static I-O analysis covers the above mentioned technical coefficients matrix (**A**). The attention in dynamic models partially focuses on the dataset illustrating investments into technical capital. Therefore, I expand the basic model for the fixed capital stock matrix (matrix **F**) and coefficients of capital intensity (matrix **B**). Matrix $\mathbf{I}^F$ is the next considered matrix depicting the difference of matrix **F** elements in time *t+1* and in time *t*. I will discuss these matrices in more detail in chapter 2.3 of this paper.

Dynamic I-O models are characteristic for their endogenization of exogenous variables of the static I-O model. Fundamental dynamic model endogenizes the above-discussed influence of investments into technical capital. In contrast to the static version, such information now enters the production equations of the model itself. More complex models then endogenize wider scale of parameters, which should assist analysis of monetary and fiscal effects.

Significant attribute of dynamic I-O models is their transition from purely structural models to structural-growth models due to their extension for investments into technical capital. Eurostat (Eurostat, 2008, p. 517) denotes these models as “multiplier-accelerator models”. These models serve especially for an examination of structural relationships within an economy. Next, they also embrace different long-run relationship between industries deforming own structure of an economy as defined in the model. This model should then result in more exact outcome in comparison to the basic static model, which neglects these factors.

Dynamic models generally expand and optionally modify elementary set of static I-O assumptions. Laščiak (1985, p. 132) states the principal assumptions as:

- Linear relationship between coefficients.
- Each industry has a firmly defined structure of inputs.
- No option of substitution between inputs and outputs.
- Capacity and work norms are exogenously defined.
- The model abstracts from the influence of foreign trade.
- Final consumption is generated as a residual variable.
- Entrance of all inputs of the model is smooth and continuous.
- The model is built upon a uniform cost structure or structure of production.

Fecanin (1985, p. 64) completes these assumptions for:

- an assumption of a unique source of production capacity growth in a form of investments into technical fixed capital;
- calculation of investments into a non-profit sector in final consumption matrix instead of investment matrix. This notion depends on a particular model and approach; it only pertains to basic dynamic model.

I can therefore define the basic dynamic I-O model as one encompassing direct but also indirect links between production and capital (see Goga, 2009, p. 103).

It is noteworthy to mention the absence of impact of the basic dynamic Input-Output model on the structure of simulated economy; it does not modify the ratio between inputs and outputs of individual industries. Hence, the model does not reflect the possible variability of multipliers caused by alteration of matrix **A** structure.

The dynamic model can be represented with help of difference as well as differential equations. Production function of the model appears as:

$$\sum x_i(t) = \sum\sum a_{ij} x_j(t) + \sum\sum b_{ij}^i \left[x_j(t+1) - x_j(t)\right] + \sum y(t). \tag{6}$$

In a matrix form (Eurostat, 2008, p. 517):

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\left[\mathbf{x}(t+1) - \mathbf{x}(t)\right] + \mathbf{y}(t). \tag{7}$$

The resulting form of the model is:

$$\mathbf{x}(t+1) = \mathbf{B}^{-1}\left[\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}(t) - \mathbf{y}(t)\right]. \tag{8}$$

To compare such equation with the classical Input-Output model:

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}. \tag{9}$$

Closed I-O model takes a form in case of difference equations:

$$\mathbf{B}\mathbf{x}(t+1) - (\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}(t) = 0. \tag{10}$$

Final set of equations in case of differential equations (For proof, see: Diaz, Carvajal, 2002):

$$x(t) = e^{Mt}X(0) + \int e^{M(T-\tau)}NY(\tau)\delta(\tau). \tag{11}$$

The last equation reflects the equality between the total volume of capital used in future year and contemporary volume of capital plus final consumption. The investments into fixed capital are endogenized in this closed model with help of discontinuous model on the basis of differences. Final consumption stays as the only unknown and exogenous variable (except constant). As already mentioned, the final consumption variable is often modeled as a residual variable or with a use of a specific utility function.

2.2 Technical Capital Matrix

The main part of the dynamic Input-Output model consists of the matrix **B**. This matrix is a capital-intensity coefficients matrix or the capital stock matrix as already mentioned. This matrix depicts a quantity of technical capital produced in time *t* and employed in time *t+1* (Leontief, 2007a, p. 295, p. 316) calls this matrix "Capital stock coefficients"; matrix which depicts states of technical capital. The matrix is then supposed to encompass all capital flows into particular industry in individual years controlled for depreciation.

It is possible to interpret the meaning and definition of this matrix differently. A classical definition assumes the elements of the matrix to illustrate the ratio of quantity of capital stock delivered from industry *i* necessary for production of one unit of output in industry *j*. Following this definition, one might express this matrix as:

$$\text{Matrix: } \mathbf{B}^F = (b_{ij})_{n\times n}, \qquad \text{with elements: } b_{ij} = \frac{f_{ij}}{x_j}. \tag{12}$$

That is ratio between all capital flows from industry *i* to industry *j* designated for one unit of production in industry *j*. For all coefficients relation $0 \le b_{ij}$ must stay true in this definition of matrix, as industries cannot keep negative amount of capital.

One can also define the matrix **B** as investment coefficients matrix (Goga, 2011). Therefore, the construction of the matrix utilizes on the relationship about the long-run stability of technical coefficients of the matrix **B** and on the equations of the dynamic model (you can find the proof below). Keeping these relations in mind one can calculate the matrix as:

$$\text{Matrix: } \mathbf{B}^I = (b_{ij})_{n\times n}, \qquad \text{with elements: } b_{ij} = \frac{i_{ij}}{\Delta x_j}. \tag{13}$$

The elements are hence calculated as the ratio of capital flow in one year (new investment) and production change. However, this computation method does not guarantee the validity of previous limitations about their non-negativity. In this case, investments can be negative due to depreciation, the production difference might appear below zero and investments may grow. For this reason, negative coefficients are transformed to null ones. This way defined matrix does not fully correspond to the basic dynamic model in its meaning. It rather artificially sets states of capital according to the capital flow in one year (investment) and to production changes dependent on this alternation.

In respect to the equations one can find several conditions, data and model must meet while using the computation method for the matrices **B** and **F**:

1. Change in capital variable (investment) tends to long-run equilibrium even in the short-run and hence does not undergo any long-run volatility; the same stays true for investment and production.

Taking into consideration the relation, $f_{ij} = \frac{i_{ij}}{\Delta x_j} x_j$, one can deduce proportional influence of the change in investment or production on the total stock of capital accumulated in the previous periods of time.

2. Previous-year or older effect of investment changes does not affect the present change in production; otherwise the total stock of technical capital would alter.
3. The entire effect of investment adjustment is carried on this year, as the technical capital would alter otherwise.
4. Situation where a industry does not invest into a specific flow of capital (*i*, *j*) must not occur even if it satisfies the above stated conditions and thus confirms the principle of linearity of relationships between investment and production changes. The reason is the zero value of estimated stock of capital obtained by this method (under the assumption that the capital is being used) regardless the capital accumulation recorded in the previous year.
5. Situation when production of a sec equals zero must not occur since the equations would not have solution.

These conditions are rather strict for calculation of necessary matrices. Most probably the outcome is not going to be robust from the standpoint of long-run stability of coefficients. In the short-run coefficients might appear volatile, one can assume delayed effect of investments and finally, the effect of investment adjustments from previous years might affect the output within the observed period of time. It is not possible to fully eliminate such problem but it is possible to minimize it by applying calculations based on longer time series with a consideration of an existence of depreciation. This idea will be discussed in section 2.3.1 in bigger detail.

Matrix $\mathbf{B}^F$ appears from the calculation perspective as more robust than matrix $\mathbf{B}^I$. One of my arguments is a possibility of significant differences between matrix $\mathbf{B}^I$ and matrix $\mathbf{B}^F$ in a reaction to production and investment volatility within one year. Then this relation seems as more reasonable:

$$\lim_{t \to \infty} b_{ij}^I \approx b_{ij}^F . \tag{14}$$

It depicts the raise in approximation of elements of the matrix $\mathbf{B}^I$ to elements of matrix $\mathbf{B}^F$ in long-run.

Last but not least, this analysis might be accompanied with a problem of singularity of matrices. Since not all industries produce technical capital, the matrix **B** is practically always singular. This characteristic aggravates the utilization of dynamic Input-Output models. One can partially but not absolutely evade this obstacle by applying pseudo-inverse methods as already mentioned. According to many results (see Miller, Blair, 2009; Šafr, 2014), such modification of model with pseudo-inversion leads to unstable estimations, which inclines to "exponential" growth. Such outcome also results in low values of the matrix **F**.

2.3 Construction of Capital Stock Matrix (B)

2.3.1 Extrapolated (model) approach

There are several methods how to construct a capital stock matrix. Common diversification understands direct methods based on primary collection of data and indirect methods. The latter types of methods derive the capital matrix from other sources of data or from direct calculation from the model. The Czech Statistical Office does not publish capital flows/stock matrix classified by industry or production. For this reason, I need to find a different way of dataset collection for the matrix, for example its calculation from other sources of data.

First and second presented method for matrix computation is based on detail knowledge of production allocation for GFCF (non-symmetric matrix). I am going to assemble a symmetric capital flow matrix within one year using symmetrizing methods for the matrix "product x industry". These methods have been originally derived for symmetrizing SIOT tables. This part is common for both methods.

Dynamic Input-Output models can serve for a calculation of matrix **B** by the approximation procedure (Goga, 2011, p. 81). The core assumption of I-O models understands production function as constant in the long run. Mathematical form of this assumption:

$$f_j^t(x_j^t) = f_j^{t+p}(x_j^t). \tag{15}$$

Application of this assumption for dynamic Input-Output models especially for the capital matrix could be written as (Eurostat, 2008, p. 520):

$$\mathbf{I}^F(t) = \mathbf{BX}(t+1) - \mathbf{BX}(t). \tag{16}$$

This relation displays the investment matrix in time t as the difference between production-flow matrix multiplied by capital coefficients in time $t+1$ and the same matrix in time t. Then this equation must stay true (similarly as for the technical coefficients matrix $\boldsymbol{A}$):

$$\frac{f_{ij}(t)}{x_j(t)} = \frac{f_{ij}(t+p)}{x_j(t+p)}. \tag{17}$$

The above equality states coefficients of matrix **B** as stable and constant in the long run. One might obtain the same results by using dataset in time t, $t+1$ or $t+p$. If this assumption holds then:

$$f_{ij}(t) = \frac{f_{ij}(t)}{x_j(t)} x_j(t), \qquad f_{ij}(t+p) = \frac{f_{ij}(t)}{x_j(t)} x_j(t+p), \tag{18}$$

$$f_{ij}(t) = \frac{f_{ij}(t+p)}{x_j(t+p)} x_j(t), \qquad f_{ij}(t+p) = \frac{f_{ij}(t+p)}{x_j(t+p)} x_j(t+p),$$

These equations can be applicable for construction of GFCF matrix or investment matrix $\mathbf{I}^F$. The above-mentioned formula (18) stays valid for this matrix (**F**). Using **previous formulas** (18) along with the formulas (15) and (17) one can obtain these relations:

$$\sum_t^{t+p} i_{ij}(t) d_{ij}(p) = f_{ij}(t+p) - f_{ij}(t), \tag{19}$$

then:

$$f_{ij}(t+p) - f_{ij}(t) = \frac{f_{ij}(t+p)}{x_j(t+p)} x_j(t+p) - \frac{f_{ij}(t)}{x_j(t)} x_j(t), \tag{20}$$

modified to:

$$\sum_t^{t+p} i_{ij}(t) d_{ij}(p) = \frac{f_{ij}(t)}{x_j(t)} \left[x_j(t+p) - x_j(t)\right], \tag{21}$$

where I substitute:

$$\Delta x_j^{p,t} = x_j(t+p) - x_j(t), \tag{22}$$

to obtain:

$$\sum_{t}^{t+p} i_{ij}(t) d_{ij}(p) = \frac{f_{ij}(t)}{x_j(t)} \Delta x_j^{p,t} . \tag{23}$$

Our aim is to express capital value (**F**) and coefficients of matrix **B**, therefore:

$$f_{ij}(t) = x_j(t)\left(\Delta x_j^{p,t}\right)^{-1}\left[\sum_{t}^{t+p} i_{ij}(t) d_{ij}(p)\right], \tag{24}$$

and

$$b_{ij}(t) = \left(\Delta x_j^{p,t}\right)^{-1}\left[\sum_{t}^{t+p} i_{ij}(t) d_{ij}(p)\right], \tag{25}$$

where:

Matrix: $\mathbf{B}(t) = \left(b_{ij}(t)\right)_{n\times n}$, where: $b_{ij}(t) \approx \dfrac{f_{ij}(t)}{x_j(t)} \approx \dfrac{f_{ij}(t+p)}{x_j(t+p)}$, (26)

$0 \le b_{ij}$, $i, j = 1,2,\ldots\ldots, n.$

For matrix of flows:

Matrix: $\mathbf{F}(t) = \left(f_{ij}(t)\right)_{n\times n}$, $i, j = 1,2,\ldots\ldots, n.$ (27)

and investment:

Matrix: $\mathbf{I}^F(t) = \left(\sum_{t}^{t+p} i_{ij}(t) d_{ij}(p)\right)_{n\times n}$, $i, j = 1,2,\ldots\ldots, n.$ (28)

Aggregate form of these two outcomes are:

$$\mathbf{F} = \mathbf{I}^F \overline{\left(\Delta\mathbf{x}(t)\right)}^{-1} \overline{\left(\mathbf{x}(t)\right)} , \tag{29}$$

$$\mathbf{B} = \mathbf{I}^F \overline{\left(\Delta\mathbf{x}(t)\right)}^{-1}, \tag{30}$$

where:

$\overline{\left(\Delta\mathbf{x}(t)\right)}^{-1}$ – Inverse matrix with diagonal elements of newly created investments.

$\overline{\left(\mathbf{x}(t)\right)}$ – Matrix, which diagonal elements are vector of total production in time *t*.

This model procedure of computation of technical capital matrix might carry several already discussed problems. For this reason, I derive general solution for longer time period:

$$f_{ij}(t) = x_j(t)\left(x_j(t+p) - x_j(t)\right)^{-1}\left[\sum_{t}^{t+p} i_{ij}(t) d_{ij}(p)\right]. \tag{31}$$

Analogically for time t + p:

$$f_{ij}(t+p) = x_j(t+p)\left(x_j(t+p) - x_j(t)\right)^{-1}\left[\sum_{t}^{t+p} i_{ij}(t) d_{ij}(p)\right], \tag{32}$$

where $d_{ij}(p)$ represents depreciation rate of investment $i_{ij}(t + p)$ in time p. This depreciation is determined for capital type (i) as well as for industries (j) employing the capital in specific time ($t + p$).

For time period p, which does not have to be necessarily continuous from the perspective of stability of technical coefficients, implementation of this way of calculation yields $N_{F/B}$ various ways of solutions for matrices **F** and **B**, where:

$$N_{F/B} = (p-1)p. \tag{33}$$

The final averaged matrix $\overline{\mathbf{B}}$ can be obtained by calculating averages for individual matrices:

$$\overline{\mathbf{B}} = \left(\overline{b}_{ij}\right)_{n\times n}, \qquad \overline{b}_{ij} = \sum_{w=1}^{(p-1)p} \frac{b_{ij}^{w}}{(p-1)p}. \tag{34}$$

Resulting matrix $\overline{\mathbf{B}}$ symbolizes extrapolated solution. It is an averaged technical coefficients matrix.

Non-solved question arises, namely if such computed matrix is in accordance with the original theory. The construction way of the matrix classifies it as rather matrix of willingness to invest (Díaz, Carvajal, 2002, p. 12), which could be viewed as positive from the standpoint of construction of a dynamic model.

I expect this matrix to significantly vary from the matrix formulated with the second method. However, this matrix should include all effects potentially affecting the matrix of fixed states not necessarily apparent in statistics such as faster capital depreciation.

2.3.2 Construction of the CPA matrix from the AN matrix (AN approach)

Second method utilized in the application part to compose the matrix **B** originates in the available capital state matrix according to the classification AN x NACE. First, transformation of the AN matrix to CZ-CPA takes place. The conversion procedure employs knowledge about the GFCF matrix (in classification CZ-CPA x NACE) along with suitable interference concerning individual flows.

The total value of capital states is based on the realization of the matrix AN x CZ-CPA. Following this classification, the resulting CZ-CPA x NACE matrix is symmetrized in accordance with the method “A“ in the manual of Eurostat (2008). To be precise, the procedure is not matrix symmetrization by definition but transformation to CPA x CPA classification, as it does not have to yield equality between the sum of values of columns and rows.

2.4 Complete Final model

The outline of the complete final model enters the matrix form as:

$$\mathbf{x}(t+1) = \mathbf{B}^{-1}\left[(\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}(t) - \mathbf{y}(t)\right], \tag{35}$$

$$\mathbf{y}(t+1) = (\mathbf{A} - \mathbf{I})\mathbf{B}^{-1}\left[(\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}(t) - \mathbf{y}(t)\right], \tag{36}$$

$$\mathbf{F}(t+1) = \left[(\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}(t) - \mathbf{y}(t)\right], \tag{37}$$

$$\mathbf{I}(t+1) = (\mathbf{I} - \mathbf{A} + \mathbf{B})(\mathbf{x}(t) - \mathbf{x}(t-1)) - \mathbf{y}(t) + \mathbf{y}(t-1), \tag{38}$$

The common recommendation to avoid unstable fluctuations of resulting aggregates is an inclusion of an investment variable into a total demand (as consumption and export) and modeling it along with its outcome as a stable element. Increasing/decreasing capacities of dynamic Input-Output model then endeavors to reflect upon the varying final demand of an economy (Eurostat, p. 523).

I will replace final consumption of the model with real consumption to verify defined hypothesis about prediction capability of the model concerning production. Next, since the model does not specify calculation procedure for capital/capital coefficients in period t=0, I will apply the two above elucidated methods and compare them in the concluding ex-post/ex/ante analysis. However, regarding time period t>0 I will utilize the above described equations. In other words, the model will be calibrated with regards to real economic variables in time t=0. Production and capital estimated for subsequent time period will initiate final consumption, in which most solutions of other models (Idenburg, Wilting, 2000) vary. Math representation:

$$\mathbf{x}(t+p+1)=f\left(\mathbf{x}(t+p),\mathbf{A}(t),\mathbf{B}(t),\mathbf{I}(t+p),\mathbf{y}(t+p+1)\right). \tag{39}$$

Differences between predicted and real future total production in time t+p+1 should primarily originate in production functions. For this reason, this study abstracts from the issue of utility function. Potential inclusion of utility function could conceal and undervalue/overvalue real prediction capability of those production functions. Finally, I assume stable relationship between production growth and gross fixed capital formation such as for example Eurostat (2008, p. 523).

3 DATA

Dataset used for the computation of the model comes from the Czech Statistical Office. This institution publishes symmetric Input-Output tables in a five-year period; the most recent SIOT table is for year 2010. I also utilize ESA 95 valid for the 31st of December 2013, in regard to the structure and data classification for capital. To calculate coefficients of the matrix **A** I work with dataset for the year 2010. The technical capital matrix is based on the time span 2005–2010 (n = 5) except for the year 2009. Exclusion of the latter year stems from my aim to minimize the bias of the model due to the economic crisis.

I compose the GFCF matrix in correspondence to the matrix product x industries (CPA x NACE) within time period of one year. I treat this non-symmetric matrix for price level changes and then symmetrize both matrices for individual years according to the structure of the use of intermediate consumption matrix. This symmetrizing part of the methodology seems to be the most simplifying and problematic. In reality, the structure of capital utilization does not probably correspond to that of intermediate consumption. Nevertheless, this procedure appears to be the only contemporary way of the matrix calculation without necessity to construct new dataset.

The resulting GFCF matrix reveals to be for both methods irregular. For this reason, I apply pseudo-inverse method to be able to solve the equations (for more information see: Jódar, Merello, 2010). The final dynamic Input-Output model reflects depreciation of gross fixed capital. The value of the capital depreciation is consistent with depreciation rates of the Czech Statistical Office commonly used for calculating the stock of GFCF.

I consider several different views of depreciation. First, I scrutinize the longevity of the capital employment; second the capital type and third the place of its depreciation. Therefore, not only various types of capital generate different depreciation, one type of capital in different industry shows to have a different length of its utilization/depreciation as well. Then the final depreciation parameters matrix yields three parameters – WHEN, WHICH and WHERE capital depreciates. Data for final consumption and GFCF are collected from the website of the Czech Statistical Office.

Model and prediction respect the classification to 82 products. For the purpose of analysis, acquired outputs aggregates to the top CZ-CPA categories.

4 RESULTS

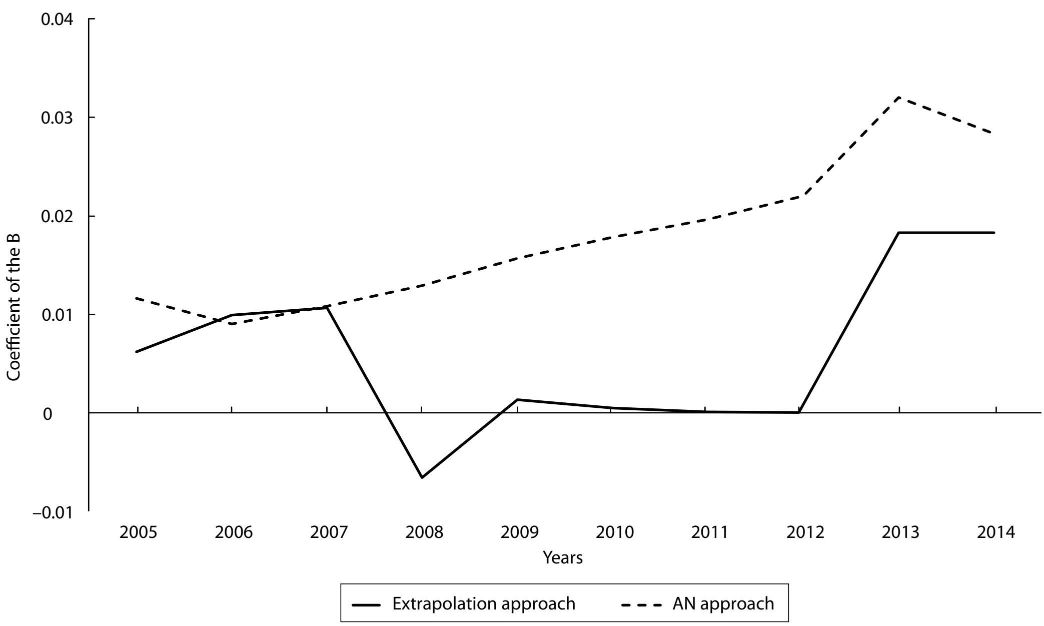
Case study of the Czech Republic for years 2005 to 2010 starts with an estimation of missing data, paying attention to the technical coefficients matrix and the capital state matrix. With respect to the available dataset I construct two solutions. The first solution (extrapolated matrices **B** and **F**) exploits the above-derived procedure for the composition of the capital matrices based on the assumption about validity of the dynamic I-O model. The second solution, the transformation of AN x NACE matrix to CPA x CPA, derives from statistically determined dataset and its manipulation to obtain the CPA x CPA matrix.

After the estimation of missing data I choose the year 2010 to serve as a base year. Next, I predict future values of production and GDP for time period 2010 to 2014 and past values for years 2005 to 2009. Regarding the base year, the minimum errors in the structure of prediction will exist in the vicinit of the year 2010.

4.1 Matrices

The coefficients of the matrix **B** as well as the matrix **A** are assumed to be stable in the long-run. The subsequent graph depicts the evolution of the median non-zero coefficients of the matrix **B**. The dashed line illustrates these coefficients obtained by extrapolation and dotted line stands for coefficients from the classification AN (see Figure 1).

Figure 1 Median non-zero coefficients of B



Source: Author's calculations

One can notice significant instability of the coefficients in the matrix **B** calculated by the method of extrapolation. The drop in the year 2008 reflects the violation of the model assumptions as a result of the financial crisis.

The next table displays the values from the extrapolated calculation divided for the two time periods before and after the critical year 2008 as well as for the entire observation period:

Table 1 Statistics of extrapolated calculation

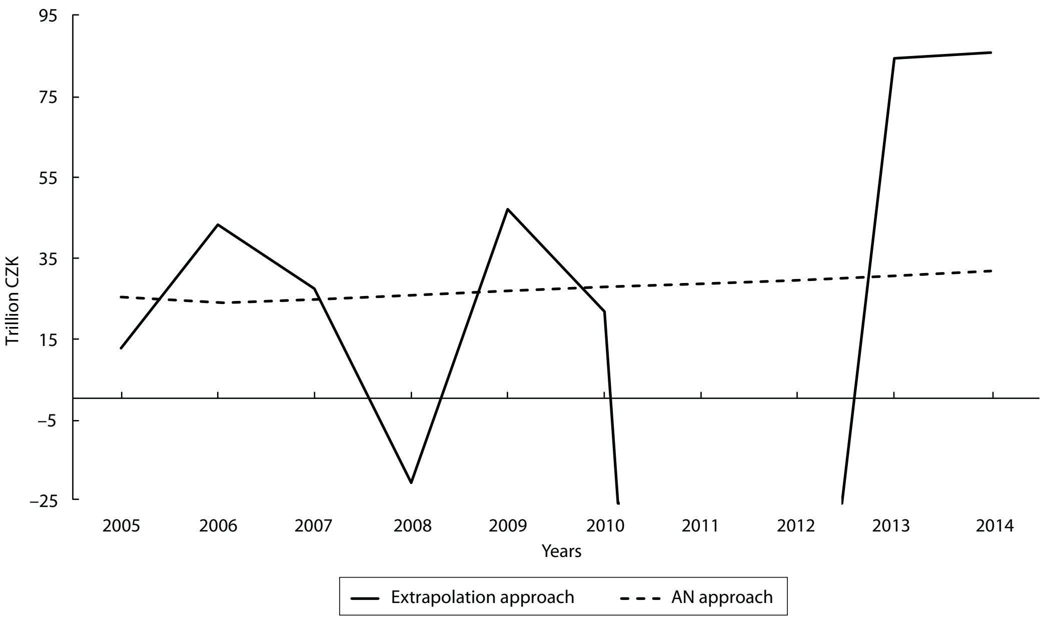
Time period	Average coefficient	% of negative values in matrix B	Average Coefficient of Variation of coefficient matrix B
2005–2013	–0.1384	5.19%	–34.56%
2005–2008	+0.0073	5.03%	+18.44%
2009–2013	–0.2551	5.31%	+836.07%

Source: Author's calculations

Negative values of the matrix **B** were transferred to null ones by applying method RAS.

Regarding the extrapolated solution, the construction of the matrix **B** precedes that of the matrix **F**. In respect to the second composition method based on the AN classification, the calculation of the matrix **B** follows that of the matrix **F** (see Figure 2).

Figure 2 Amount of fixed capital in the economy (trillion CZK)



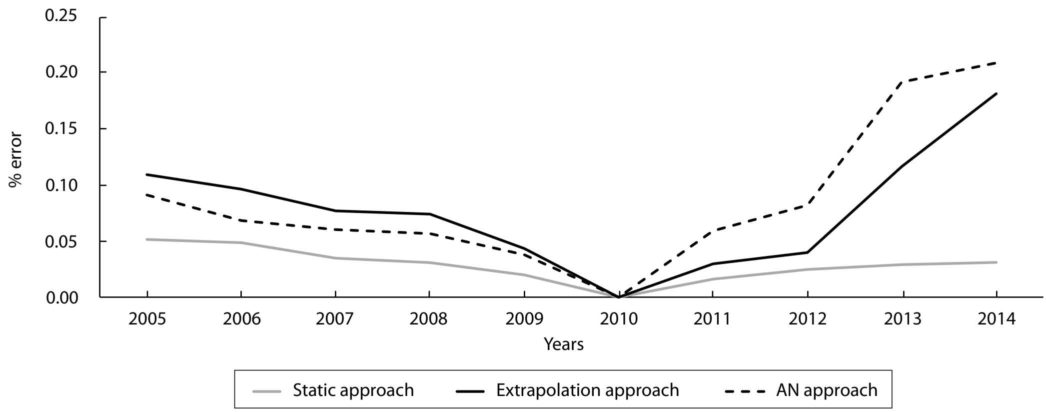
Source: Author's calculations

The volume of the stock derived from the AN classification stands for the statistically measured stock in the economy. In contrast, the stock calculated by the extrapolated method demonstrates the volatility of the model estimation caused by a violation of the assumption essential for the applied method of extrapolation. The extrapolated solution for the matrix **B** but also for the matrix **F** reflects the effect of crisis. The turmoil period is especially apparent in the growth of the negative changes in values of production. The most accurate calculation of the value of capital states occurs in the extrapolated solution between years 2005 and 2007. During this time period the capital reached on average 20.37% of the real value of measured capital.

4.2 Predictions

Dynamic Input-Output models originate in the structurally analytical models with a potential for product enhancement as an outcome of investment. Since these models are primarily structural, I start with a comparison of deviations from the predicted structure of production at the Figure 3.

Figure 3 The gradual growth of the error (%)

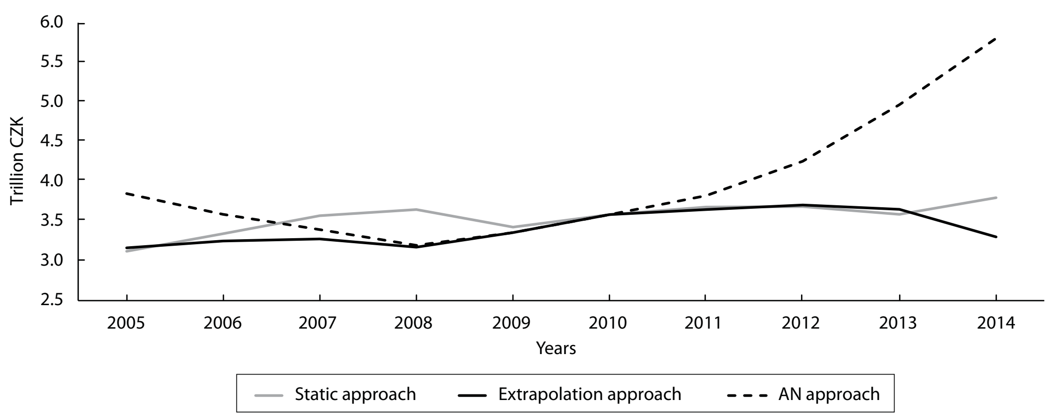


Source: Author's calculations

This graph illustrates a gradual growth of the error in the structure of prediction for the total production since 2010. The static solution exhibits a minimum deviation concerning the values before and after the year 2010. The cumulated error in the structure of prediction for the static solution reaches the value of 5.16% for the entire observation period and 3.11% if one regards only the future values. This outcome also reflects the alteration of the structure of SIOT tables with a violation of linear relationships in these tables. Extrapolated solution indicates 10.95% of error for the past values and 18.18% for the future values. On the contrary, the solution of the matrix computation from the AN classification exhibits error of 9.13% for the past and 20.92% for the future values.

The resulting prediction of gross value added (GVA) would according to dynamic and static I-O model appear between years 2005 and 2014 is at Figure 4.

Figure 4 The prediction of gross value added (trillion CZK)



Source: Author's calculations

It is worth noting the absence of real measured evolution of GVA for domestic production on this graph. The reason for such deficiency is the lack of GVA data; they are published in five-year period therefore regarding the observation period, data only for years 2005 and 2010 are present.

This graph depicts a prognosis of production drop during the period of crisis (most apparent for year 2008) for both dynamic variants. The line standing for the extrapolated solution is also decreasing. The downward trend is a consequence of the above-mentioned troublesome computational procedure concerning the capital stock, which was calculated with help of production and capital changes from one year to another. Considering the line calculated from the AN x NACE matrix, such prognosis during financial crisis reflects a reduction of specific type of investment during these years. Static model delays the reduction prognosis till 2009 as a consequence of consumption diminution. The forecast of the evolution for the extrapolated and static version assimilates development of GVA. In contrast, the method from the AN assumes a non-realistic high growth of values for the period 2010–2014.

5 DISCUSSION

The above-mentioned results signal numerous obstacles. We can categorize these problems into the question of calculation to complete the missing dataset and the issue of calculation of the respective prediction.

Both computation methods of matrices approach the matrices **B** and **F** from a different point of view. Extrapolated calculation proceeds from the matrix **B** to the matrix **F** while the calculation based on the classification AN starts with the matrix **F** to derive the matrix **B**. This seemingly trivial fact leads to the raise of cumulated errors mostly apparent in the matrix **F** when using the extrapolated calculation and in the matrix **B** in case of the calculation from the classification AN. Absolute level of capital evaluated from the extrapolated solution is then significantly lower than if one applies the calculation from AN. The capital state matrix produces significant unstable estimations of the aggregate stock of capital **F** caused mostly due to the violation of the assumption about non-negativity (or rather same direction) of the production in respect to the investment in GFCF. On the other hand, the matrix **B** indicates substantially lower variability in case of the extrapolated solution. Nevertheless, the elements of the matrix **B** are also negative for the extrapolated solution. Most authors agree upon the importance of non-negativity of the matrix **B**. Despite this concurrence, there exists an economic explanation of this phenomenon.

Next, the computation of the prediction also exhibits the inversion problem regarding the singular matrix **B**. The main difficulty occurs in the values of the matrix approximating zero and thus leading to exponential trends. Other authors indicate this problem (Miller, Blair, 2009). Some estimated predictions of product as well as GVA illustrate divergent oscillations even for such short time period.

Minimum deviations in the structure of industrial prediction occur around the year 2010. This outcome is given by a rather slower adjustment in the structure of national economy than the model assumes as well as by an absence of some effects and differences caused by a capacity enlargement, which are expected in the model for both dynamic versions. Divergence of all models from their real counterparts grows over time. The disparity between the static variant and reality rests entirely on the alteration of the structure of national economy since 2010. The static model does not reflect changes in the structure and wholly neglects the influence of capital.

The outcome of the product and GVA predictions stay consistent with the previous research especially concerning the problematic utilization of the capital matrix (Miller, Blair, 2009). Unsolved question regards the content of the matrix **B**, respectively **F**. The neglect of assets and their real effectiveness generates substantial differences between the theoretical and the extrapolated stock of capital (Diaz, Carvajal, 2002).

Furthermore, the model contains an assumption about the full impact of investment taking place within one time period, which might not necessary come true. In that case, total production becomes overestimated and the structure of respective production modifies faster. Some succeeding models eliminate this problem and exhibit better results than the dynamic I-O model alone (Mönnig, 2012). We could indisputably classify INFORUM models as such types of successful models, which combine features of a dynamic I-O model with models on the basis of CGE or other endogenous and exogenous information (Zhang, 2000; Liew, 2000).

CONCLUSION

This article aims to test the applicability of dynamic I-O models for the Czech data. Despite the attainment of construction of the respective dynamic model, range of problems and unsolved difficulties arises. Regarding absence of complete datasets, this paper shows possible construction procedures of unavailable matrices. However, both above discussed and applied ways face computation problems. Applying extrapolated solution often leads to the violation of assumptions crucial for this calculation method. The obstacle of the second approach stems from the troublesome transformation of the classification from AN to CPA. Next, utilization of the latter method necessitates a difficult decision about the content of such matrix. The particular calculation of the prediction faces problem of irregularity of capital matrices as well as display of oscillatory and exponential evolution. In the end, the resulting prognosis distinctly varies from the statistically measured real values especially for the structure of production in national economy. Namely, there appear to be eleven to twenty-one per cent erroneously allocated values during five periods. This inaccuracy might originate in the expectations integrated in the model about the impact of all effects on the economy within one year, which might not necessarily correspond to reality. Next, the model assumes the same effect to result from different types of capital. Finally, utilization of statistically computed depreciation in the model might not be truly consistent with reality.

Most of the outlined problems could be overcome by an incorporation of the production equations into an enhanced model framework. Examples of those models are based on the models such as INFORUM or DSGE. These models constitute a complex infrastructure possibly defining restricted production functions along with possible stickiness.

The subsequent research could focus on an inclusion of these production functions into a more complex economic model based on the contemporary economic theory. Next, following analysis could more thoroughly explain the phenomenon surrounding capital matrices. Difficulties concerning diverging structure over time could be explained by the steadiness and different effectiveness of capital. Finally, other endogenous variables such as employment could be used to enhance the presented model along with an inclusion of individual utility functions for various economic factors such as households or government.

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Statistical Measurement of Pension Entitlements

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Abstract

The paper deals with macroeconomic estimates of the impact of population ageing. It shows that the optimal choice of pension system is not a crucial issue. The main emphasis should be put on demographic and economic parameters influencing significantly the share of retirement pensions in our society. Presented approach is based on so-called pension schemes expressing discounted pension entitlements to 2014. It shows the liabilities to current and future pensioners for both men and women aged over 20 years. We present this model in three variants and illustrate its sensitivity to input parameters. This approach allows us to estimate the share of pension liabilities in gross domestic product that accounts for about 230% according to generally recommended approach. Pension entitlements represent a modern topic to be implemented in 2017 in national accounts. Pros and cons of this modern approach are discussed within our paper.

Keywords	***JEL code***
Ageing of population, national accounts, pension liabilities, pension schemes	*E17, H55, J14*

INTRODUCTION

Population ageing is very popular topic in the media even though interpretation and consequences are not sufficiently discussed. The substance of this issue is very often deferred in favour of political or interest groups' preferences. The presentation of the pension systems problem is usually very simplified and key issues are hidden behind. The emphasis is put on the choice of an optimal pension system and its possibilities to satisfy pensioners' needs. But the key problem of this issue lies in deeper elaboration of the situation. Detailed analyses or experts' judgements are not usually widely presented to public. Discussion about stable, sustainable and usable general approach to the population ageing is missing.

Mostly, the discussion is focused on the choice of pension system. Economic fundaments relying on share of economically active population, fertility, economic growth and productivity are not highlighted. Optimal choice is only a particular issue of the solution of the population ageing problem. It is rather connected with the mix of solidarity and merits than economic solution. On the contrary, frequent changes of pension systems do not provide stabile economic environment and it means that the main features of the system need general political support.

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Crucial aspect of population ageing lies in the liabilities of active population to the pensioners. The obligation represents sustainability of the system ensuring pensioners' needs. Statistical estimates of these liabilities can be expressed by pension entitlements covering obligations towards current and future pensioners. It is a modern approach combining national accounts' and demographic data. On this case we illustrate that the choice of the pension system is not very relevant issue from macroeconomic perspective. Both Pay as You Go pension plan (PAYGO) and Fully Funded Pension (FFP) express finally the same. Contributions or benefits defined systems represent redistribution of products between active and inactive population. Generally, it means that pensioners will consume existing goods and services that have to be produced. The liabilities to pensioners are equal irrespective of the selected pension system. The presentation of pension entitlements is expressed as a share of gross domestic product representing the amount of goods and services to be transferred to pensioners. Such information should be used by policy makers to adopt economic environment for future demographic development.

This paper presents our provisional estimates of pension entitlements from the perspective of statisticians. Czech national accounts will include the estimates of pension entitlements in 2017. Presented results come from our computation and are based on slightly simplified approach in comparison with official statistics. The aim of our paper consists in the illustration of significantly different approach to pension liabilities.

1 ECONOMIC APPROACH TO POPULATION AGEING AND PENSIONS

The discussion about economic aspects of ageing and the necessity of suitable pension system choice has been discussed in scientific literature for several years. The issue of the choice of pension system is discussed in Vavrejnová et al. (2004). It also proves that the issue of pension reform should not be limited only to ageing and financial matters. All pension systems, both private and public, have their own pros and cons. Combination of different pension benefits, individual savings and prolonged economic activity is presented as an optimal choice. The key target is to ensure adequate living standards and quality of life of elders.

Sivák et al. (2011) described consequences of transition from PAYGO to fund based systems in Slovakia. They emphasised findings that each system is based on the distribution of real income from productive generation to non-productive. Retirees can consume only from produced goods and services in each period. Transition from one system to another leaves aside the main point, demographic situation. The overall increase of wealth of society is not changed. The only difference consists in the different transfer of income between productive and non-productive population.

According to Kubíček (2008) it is not important whether the system is fully or partially private. Again, this question does not tackle the substance, demographic development of our society. It seems that the only possible solution could be found in imposing new taxes to enable sufficient resources. This question of raising taxes is currently discussed also in both media and expert groups.[3] The necessity of pension reform was widely analysed in Rutarová and Slavík (2005). The reform was discussed from the perspective of both economic and demographic point of view. It was emphasised that current pension issues are linked with state based PAYGO system and its affordability. It is pointed out that demographic development should not be used as an argument for the change of the pension system or its reform. Economic substance lies in the inter-generational transfers covering goods and services. Active population has to relinquish a part of its consumption in favour of pensioners. Financial transfers serve as a mean of transfers only.

3 <http://idea.cerge-ei.cz/files/IDEA_Studie_14_Zdaneni_vysokoprijmovych_osob/IDEA_ Studie_14_Zdaneni_vysokoprijmovych _osob.html#p=18>.

Fiala and Langhamrová (2015) noticed that one of the most discussed consequences of population ageing is the sustainability of pension systems. Due to the increase of life expectancy, the share of pensioners in population is rising as well as the duration of receiving social benefits. The current system of pensions is threatened by increasing difference between revenues and expenditures. They estimate that such deficit can be decreased by increasing contributions to the state pensions system (social security contribution). Similarly, Janíčko and Tsharakyan (2013) refuse the necessity of pension reform from adverse demographic development only. They argued that the necessity of pension reform should be well-funded by other aspects covering mainly economic, social and political matters. It is necessary to find such parametrical adjustment that minimise the impacts on economy. They also pointed out that the participation of elders in the labour market should be increased. It is conditioned by the adaptation of labour market to older employees.

2 MODEL OF PENSION ENTITLEMENTS SCHEMES

Statistical approach to society liabilities is monitored by so-called pension schemes. They became a part of official macroeconomic statistics for the EU (see ESA 2010) and EU countries will be obliged to compile pension entitlement models from 2017.[4] There can be found lots of alternative estimates of pension entitlements in Technical Compilation Guide (see Eurostat, 2011). It all leads to the improved balance sheet for the country. It means that the balance sheet for government sector should include a new type of liabilities.

Currently, there were published few studies dealing with the pension liabilities in European countries (Van Der Wal, 2014; Oksanen, 2004; or Mink, 2010). Since there is no official data up to now, we estimated pension entitlements for the Czech Republic. We applied the adjusted approach that slightly differs from the official statistics since our aim is to estimate overall liabilities (entitlements for the population). It means that we do not treat government employees separately and we respect the population characteristics only. We estimate pension entitlements (PE) for both current pensioners (*CPE*, formula 1) and people older 20 years – future pensioners (*FPE,* formula 2):

$$CPE_t^g = \sum_{x=d_t}^{100} \sum_{s=0}^{100-x} R_{x+s,t+s}^g \cdot B_{x+s,t+s}^g \cdot p \cdot 12 \cdot DF_{t+s}, \tag{1}$$

$$FPE_t^g = \sum_{t+1}^{t+80} \sum_{s=0}^{100-d_t} R_{d_t-s,t+s}^g \cdot B_{d_t-s,t+s}^g \cdot p \cdot 12 \cdot DF_{t+s}, \tag{2}$$

where:

$R_{x,t}$ the number of retirees in age x in the year t,
$B_{x,t}$ the average of monthly retirement pension in age x in the year t,
p index of wage growth,
DF_t discount factor in the year t,
s counting index reflecting the increase in the age cohorts as well as increase years,
t year,
x age,
g sex,
d_t retirement age in t,
x_t age in t and $x_t > d_t$.

[4] Actually, this activity may threaten due to the resistance of countries with conservative approach using benefits defined systems.

Pension entitlements are derived by applying actuarial estimation methods based on the Net Present Value (NPV) concept, see Eurostat (2011). They represent future liabilities calculated to 2014 (t_b = 2014), so it is necessary to specify the assumption about future demographic development and increase in level of prices and wages.

1. *Discount factor*. It is a tool to measure the actual capital costs for financing future payments:

$$DF_t = \frac{1}{(1+r)^{t-t_b}}, \tag{3}$$

where:

r discount rate,

t year,

t_b the base year.

The denominator of discount factor includes a discount rate, which is the measure of the discount factor level. Its choice need not to be an easy task and determining the appropriate discount rate could result in scientific work nowadays. To illustrate the method in accordance with the European Central Bank (ECB, 2010) we use the discount rate of 5% in nominal terms according to the Guide (mentioned above).

2. *Wage growth*. Assumptions about future development of wages have a significant impact on the level of pension entitlements, despite difficult predictability. Wages have risen by 5% on average in the last ten years. But after the financial crisis in 2007–2008 the wage growth slowed down to 2%. For simplicity and precaution we assume such wage growth in the future (2%). The Ministry of Labour and Social Affairs publishes the average monthly retirement pension by gender. The average retirement pension in 2013 is annually increased by 2% to 2050. We do not consider any special assumptions about required insurance duration.

3. *Inflation*. It has to be considered if future flows are to be projected in nominal or in real values. If nominal term is chosen, both the discount rate and the wage growth rate include future inflation expectations. If the projection is based on real values, inflation expectations are excluded (Eurostat, 2011). For simplicity, we use only nominal.

4. *Demographic development*. The amount of future pension liabilities depends on the future development of the survival probability and life expectancy. All these aspects lead to the increase of the proportion of elderly people in population. The survival probability determines the number of years of receiving retirement pension. The official population projection of the Czech Statistical Office[5] respects the future demographic development, see CZSO (2013). This projection is taken as the basis for the estimate of future number of people receiving retirement pensions. The calculation of pension entitlements is divided by gender because of differences between mortality rates of men and women.

5. *The age of retirement*. All estimations in this paper respect the main points of actual pension reforms, retirement age is increasing. According to the current plans, the age of retirement is going

[5] The projections are generally produced in few variants according to the different sets of fertility, mortality and migration assumptions. Projection of the CZSO has three variants – low (pessimistic), medium (realistic) and high (optimistic). In this paper we use medium variant because it is considered as the most probable. However, the results should be interpreted in the sense of defining the expected development with respect to extreme variants (CZSO, 2013).

to be increased from 62 for men and 60[6] for women with two children in 2013[7] to a common threshold of 67 years in 2041. Then, it will continue to increase without limits. In 2050, the retirement age it will be about 68 years for both men and women (ČSSZ, 2014). However, for comparison, we present also two hypothetical variants of retirement age.

Let's suppose three following variants:

VAR 1 = hypothetical situation without actual pension reform. Statutory retirement age remains at 62 years for men and 60 for women with two children. This variant allows us to estimate savings from pension reform.

VAR 2 = actually valid situation respecting actual pension reform.

VAR 3 = hypothetical situation with the increase of the retirement age to 73 years in 2050. Target value is achieved by linear extrapolation. After 2050, we fixed the retirement age at 73 years. According to our calculation, this is the age (73) maintaining the ratio between economically active persons and economically inactive persons in current proportions (2014). This balance is important for keeping stable financial burden of the pension system stable and for maintaining the satisfactory living standards of older people.

The sum of current household pension entitlements of individual ages represents the total pension entitlements to current pensioners and future pensioners to actual (base) year:

$$PE_t^g = CPE_t^g + FPE_t^g \,, \tag{4}$$

where:

CPE_t^g pension entitlements for current pensioners,

FPE_t^g pension entitlements for future pensioners.

Even though we take into account the criteria of age and gender, to the relevant criteria can be found there as well. For overall macroeconomic expression of the impact of total pension entitlements on current and future pensioners, such simplification is adequate. However, from individuals' point of view, other criteria like education (Zimmermann et al., 2014) are important. It can be proved that retirement pension will also tend to be higher for individuals with higher education.

3 THE RESULTS

The liabilities resulting from pensions are crucially dependent on age. The development of pension entitlement used for the expression of liabilities is shown in Figure 1 that describes distribution of discounted pension entitlements by age to the 2014. Cohorts at the age of retirement (between 60 and 70) have the highest claims for pensions. It is considered that these people contributed to the pension system long time. The claims of older people are decreasing in line with development of mortality. Future pensioners will spend slightly shorter time in pension due to on-going increase of retirement age in comparison with current pensioners. Firstly, future pensioners will be retired at higher age and, secondly, the absolute number of pensioners will be lower because of lower fertility. However, the main reason for lower pension entitlements of future pensioners is discounting. These entitlements are expressed at current prices.

If the retirement age is not increased and remains on 2014 level (VAR 1), pension liabilities to future pensioners will be significantly higher. For example, the difference in pension entitlements for 20 years old men takes 18.7 CZK bn. and 21.1 CZK bn. for women (VAR 2). If the retirement age is

[6] Exactly, the statutory retirement age is determined according to the year of birth. By 1.1.2013 it is set as 62 years and 6 months for men born in 1950 and 59 years and 4 months for women born in 1953 with two children.

[7] In 2016, statutory retirement age is 63 years for men and 60.3 for women with two children.

increased to 73 in 2050[8] (VAR 3), the savings between (VAR 1) and (VAR 3) is higher. In case of 20 years old men it takes 25.3 CZK bn. and 26.9 CZK bn. for women in 2014.

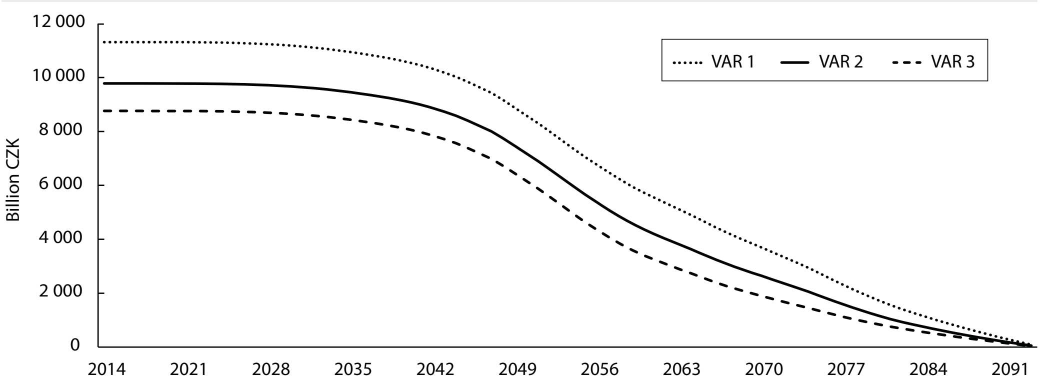
Figure 1 Pension entitlements by age in the base year (billion CZK)

Source: Data CZSO, authors' calculations

Differences in pension entitlements between men and women are not significantly high. Women live longer and therefore their participation in pension system is longer but they usually earn less than men and their average retirement pension is lower. Moreover, retirement age for women will be increased faster to balance it with retirement age of men.

The development of total liabilities of society (formula 4) for retirement pensions is determined by the concept of accrued liabilities to existing and future pensioners, older than 20 years. With using ADL[9] model, total pension liabilities discounted to 2014 accounts for CZK 9 780 bn., see Figure 2. The increase of statutory retirement age to 68 years in 2050 (VAR 2) causes the decrease of liabilities to CZK 1 535 bn. in comparison with VAR 1. Setting alternative statutory retirement age to 73 years in 2050 (VAR 3) would decrease liabilities by CZK 1 020 bn. CZK (difference between VAR 2 and VAR 3).

Figure 2 Total pension entitlements discounted to 2014 to people older 20 years (billion CZK)



Source: Data CZSO, authors' calculations

8 The age of 73 years represents the situation where share of active and inactive population is in balance.
9 ADL (accrued-to-date liabilities) expresses the present value of future retirement pensions, for more detail see Eurostat (2011).

Estimation of entitlements (irrespectively financial or non-financial) shows that the problem is serious and much deeper than the discussion about pension system. The pension system is only a part of the problem and it is rather more connected with personal motivation, equality and responsibility than with macroeconomic stability. The key problems lie in demographic parameters. It is clear that the reduction in pension entitlements will decrease the burden and resources may be spent on different goods and services. Moreover, outstanding issue of ageing is the availability of specific services for elders or further improvements of health care systems. Financial estimates of entitlements interestingly illustrate the situation in mentioned above. Without further limitation of reduction of the share of productive (active) part population, Czech society may face significant problems. It can be assumed that economic growth and the increase of productivity may compensate it but it is not sufficient.

4 SENSITIVITY ANALYSIS OF ADL MODEL OF PENSION ENTITLEMENTS

Setting of general assumption for the estimation of entitlements is necessary otherwise even rough estimates of liabilities cannot be computed. However, these assumptions may vary in time and the total value of liabilities may significantly change. Despite the dependency on parameters, total value of liabilities represents important information (Holzmann et al., 2001).

Parameters covering discount rate, rate of growth of wages and statutory retirement age significantly influence statistical expression of liabilities. Initial parameters of our computation were set as 2% nominal wage growth, discount rate of 5%[10] and increase of statutory retirement age in line with actually implemented pension reform. With these assumptions the total amount of pension liabilities takes 230% of gross domestic product in 2014. This amount covers both existing pensions and possible future claims on pensions.

When using lower discount rate, the amount of pension liabilities is rising. On the contrary, using lower growth rate of wages leads to the decrease of pension liabilities. These parameters crucially influence the results of our estimates even though it is necessary to set them ex-ante. Eurostat Manual was prepared and issued in different economic environment that is prevails today. Recommendation on the real discount rate about 3% is not adequate today. The price of money is very low; market interest rates are sometimes negative.[11] Unfortunately, it has substantial effects on the estimates of future liabilities. Very low market interest rates and prices of low risk assets (used as a discount factor) lead to higher discounted liabilities. The dependency on discount rate is stronger than on demographic factors. It means that these computations have to be discussed very carefully; they mostly provide indicative overview of the situation. For example, 3% of discount rate increases discounted pension liabilities by 140 p.p. of GDP in 2014.

Table 1 Total pension entitlements discounted to 2014 according to level of the wage growth and discount rate

		Growth rates of wages					
		0.5%	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
Discount rate	**1%**	17 809	20 783	24 432	28 932	34 505	41 438
		418%	488%	573%	679%	810%	973%
	3%	10 650	12 055	13 738	15 764	18 215	21 195
		250%	283%	322%	370%	427%	497%
	5%	7 154	7 896	8 762	**9 780**	10 982	12 409
		168%	185%	206%	**230%**	258%	291%

Source: Data CZSO, authors' calculations

[10] We assume constant discount rate and growth of wages.

[11] <http://www.kurzy.cz/cnb/ekonomika/statistika-financnich-trhu/kapitalovy-trh/vynosy-statnich-dluhopisu/AEBA>.

Pension entitlements are also dependent on the statutory retirement age. Demographic projection prepared by the CZSO shows that in 2050 there will be 2.75 million people retired. If there was not any reform adopted, this number would be by 935 thousand people higher. Adequately, economically active populations would decrease in the same proportion. The following Table 2 describes the impact of statutory retirement age on the value of pension liabilities. When computing pension liabilities for according to adopted reform (VAR 2), total amount counts 230% of GDP (CZK 9 780 bn.) in 2014. Pension reform provided savings of 36 p.p. of GDP, compared to 266% without any (VAR 1). Alternative approach with the statutory retirement age of 73 (VAR 3) would decrease this amount to 206% of GDP in 2014.

Table 2 Total pension entitlements discounted to 2014 according to retirement age

	VAR 1	VAR 2	VAR 3
Billion CZK	11 315	9 780	8 761
	-	1 535	1 020
% from GDP	266%	230%	206%
	-	36	24

Source: Data CZSO, authors' calculations

The amount of pension liabilities is dependent on future demographic development and the distribution of law of mortality law incorporated in the CZSO's projection. The sensitivity analysis showed huge differences when using different parameters. It is clear that the primary dependency is on the discount rate due to high number of projected years. We expect that such computations will be harmonised between European countries for acceptable international comparability. High sensitivity on input parameters decreases usefulness of this model. Again, the results should be used as indicative for policy-making. For example, actually available estimates show high variance. Pension liabilities for Austria and Germany are estimated for 360%, 269% respectively. In the United Kingdom they are estimated as 360% GDP (Mink, 2010).

CONCLUSION

Population ageing is a serious problem with lots of consequences in different areas of our society. Unfortunately, it is very often limited to the discussion about pension system or its reform. Even though, the issue of funding pension system is important and relevant, it is not the core of the problem. The core is hidden in demographic structure and its development. The choice of optimal pension system is rather more connected with personal motivation, equality and responsibility than with macroeconomic stability.

The substance lies in the share of economically active and inactive people and the possibilities of society for ensuring sufficient resources. Pensioners will consume real goods and services. The main issue is to satisfy their needs irrespective of the pension model. It is not important whether the system is based on contributions or benefits organised by state or private funds. The most important is a general political agreement with ensured stability. Parameters of the system can be adjusted time to time. The impact of the choice of the system is negligible, the real transfer of goods and services only matters. Total pension liabilities are not dependent on the system; the share on the GDP is not affected.

It is obvious that statutory retirement age will have to be increased but this has some contradictions. The increase of healthy life expectancy is slower than the increase of life expectancy and it means that some people will not be able to work and they will be more or less dependent on other types of social benefits.

Economic description of population ageing via pension entitlements will be incorporated into national accounts. The balance sheet of government sector will be enlarged. The usability of this information is clear. It provides data for the policy makers that should help in adopting optimal regulations in economic and social affairs. Of course, the results have to be carefully discussed; pension entitlement models use lots of parameters and assumptions. Negative feature of these computations lie in the very high sensitivity on input parameters and variance of results. Of course, our computation was aimed at iabilities only; the issue of possible increase of assets was not taken into account. It is clear that due to the development of our society and potential increase of productivity will be relevant factors. But the main aim of our paper was to illustrate the size of the problem even though that recording of future debts to potential pensioners is always a bit speculative. Despite that, the results are alarming.

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The Public Sector in the Czech Republic in Light of the Public Choice Theory

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Abstract

The aim of this paper is to discuss pros and cons of the current ways how the role of government in the society is analysed. In our analysis, macroeconomic aggregates provided by the satellite accounts of the public sector are used as an alternative analytical tool. This data supposedly better reflects the existing range of government activities as all government controlled entities are covered. Relevant time series by the Czech Statistical Office were prolonged making the analysis of long-term trends in the public sector size possible. The results are discussed against the theoretical background of the public choice theory. It was found out that there was an obvious bias to the deficit-driven provision of the public goods reflected concurrently in the growing indebtedness. On the other hand, the share of total revenues and expenditures remains rather stable over time.

Keywords

Public sector, government, national accounts, public choice

JEL code

E60, H11, Z18

INTRODUCTION

The role of government in the society has become one of the most frequent subjects of economic research over last decades. The increasing interest reflected profound institutional changes taken place in modern economies, especially the expanding government interventionism implemented through institutions of exceptional economic power intentionally established for this purpose and rising demand for clarification of how the government actions affect the economy. However, purely theoretical explanation would be found as insufficient or inconclusive; relevant statistical data and analysis supporting given conclusions are standardly required.

A long line of studies have examined the optimal size of government in relation to the wealth creation, economic growth or inflation. These empirical studies work mainly with the general government sector whose impact on the economy is expressed as the amount of government consumption or its share on GDP. However, a number of objections can be raised against this approach whereas these objections relate to the question of measurement and to the choice of aggregates best expressing the role of government in the society.

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First, the delimitation of the general government remains still an open issue even though this question has been widely discussed. Under current circumstances, official government policy is normally performed, in part, by units still classified outside the sector of government units. Irrespective of the classification, these units, controlled by government or operating under the instructions of government bodies, put into practice government social and economic policy. In other words, these units convey the effects of government policy to the economic system. Ignoring of some of them can lead to the underestimation of the effective role of units operating in the so-called public interest.

Secondly, the government consumption can help to explain direct impact of government outlays on GDP as it is currently defined. But, overall role of government in the society would be underestimated as a larger part of government outlays has no direct impact on GDP. As shown in the text, government institutions can become more oriented on the redistribution of existing wealth rather than on productive activities themselves. If this trend exists, then the size of government will be perceived as smaller which may not correspond to reality.

In the text below, we offer an alternative view or tool to analyse the impact of the public sector using existing and estimated data. The development of the public sector in the Czech Republic will be investigated in terms of its size expressed in alternative ways. The results will be discussed against the theoretical background of economic theories dealing with the evolution of the public sector in modern democratic economies.

1 THE PUBLIC SECTOR IN ECONOMIC RESEARCH

Along with expansion of government intervention the demand for relevant information on the size of the government sector or the public sector has been rising. Data for the government sector or the public sector is provided in number of forms; however it is worth noticing that some effects of government regulations can be hardly measured as is the case of various types of regulations such as tariffs, barriers to entry into industry, etc. Thus, data usually covers only transactions and stocks in whose government or public institutions are directly involved in the accounting sense, i.e. interventions quite easily measurable.

Extensive research has been conducted with the aim to tackle the issues as an optimal size of government, the impact of government policies and the size of government on productivity, inflation and especially on economic growth. In this respect, Carr (1989) and Rao (1989) raised strong objections of the methodological nature. Main point is the way of calculation of government output. Since it is calculated as a sum of costs, there is a strong bias toward positive relationship between the size of government and economic growth. Moreover, output equalled to the sum of inputs implies that the contribution of government to total productivity cannot be effectively measured. These objections are highly relevant; the costs-based quantification of government output is, undoubtedly, a weak point in measuring of government activities.

In empirical researches, the size of government is preferably expressed as the amount of government consumption[2] or total government outlays.[3] In other words, it is based on the current way how the general government sector is identified. Nevertheless, the current classification has its own pros and cons and the on-going discussion and methodological changes in the underlying data can alter final findings of empirical researches examining the effects of government policies on the economy. Rewriting history can be to high extent prevented by using alternative tools as will be shown below.

Using these tools, we will investigate whether the development of the public sector in the Czech Republic complies with the suggestions of the public choice theory as formulated in Buchanan and Wagner (2000). In this seminal work in the field of the public choice theory, Buchanan and Wagner (2000)

[2] E.g. Barro (1989), Gunalp and Dincer (2010), Chobanov-Mladeova (2008).
[3] E.g. Gwartney, Lawson and Holcombe (1998), Arpaia and Turrini (2008).

deal with the dynamics of the public sector over time. As this theory goes, the application of Keynesian macroeconomic policy in democratic societies inevitably leads to deficit financing irrespective of business cycle phase (and debt financing instead of tax financing of government outlays), inflationary pressures caused by fiscal policy and an expansion of the public sector in terms of its share on gross national income.

With these observations made by the public choice theorists, we will examine the behaviour of the public sphere using most recent data published by the Czech Statistical Office supplemented by estimated data for the sake of time series prolongation. When analysing the issue of the public sector evolution, we will incorporate main indicators as total expenditures or revenues, total amount of assets and liabilities held by units covered by the public sector which will be concurrently expressed as shares on selected denominators.

The term "public" will be used as defined in the national accounts methodology,[4] i.e. the population of units covered not only by the general government sector, but also those units considered as non-governmental, i.e. market producers under government control classified in the corporate sectors. The methodological part of the text contains more detailed information on relevant data sources and other adjustments made for the analytical purposes.

2 METHODOLOGY

Ambiguities mentioned above, which are conceptually related to the sector classification issue, can be overcome by use of alternative data set having the form of so-called satellite accounts. The satellite accounts are clearly linked to national accounts, however, units or activities are grouped differently to satisfy specific needs. For instance, it can cover purely the activities of units engaged in research and development, agriculture or health services. Main aim of this kind of accounts is to highlight specific features of particular units or activities; however, the link of satellite account methodology to national accounting is still maintained.

As our main interest is to investigate the public sector, we are dealing with the satellite account rearranging the central sector classification so that it will cover all units intentionally implementing social and economic policy of government. This is the case of the public sector satellite account whose structure, as to delimitation of group of units concerned, is more broadly described in the following paragraphs. In the Czech Republic, the satellite account covering public units is regularly published.

However, the most recent data published by the CZSO covers fairly a short period ranging from 2011 to 2013.[5] However, the analysis of the course of the public sector in term of its size and changing role in the economy should be based on data covering a longer period. The prolongation of time series has been made using data published by the CZSO and estimating data which cannot be directly overtaken from the published data. This procedure follows the sectoral structure of national accounts and the content of the public sector.

It is worthy to start the analysis with the discussion on the coverage of economic units. The public sector consists of all resident units controlled by government irrespective of market or non-market output they produce. The notion of control is defined "as the ability to determine general corporate policy of the corporation" (SNA 2008, par. 4.77). It implies that the public sector includes all units classified as government units whose economical behaviour is described by data for the general government sector (S13). Nevertheless, the methodology of the satellite accounts of the public sector goes far beyond the general government sector; economic results of market nonfinancial producers under control of government are counted in as well. Macroeconomic indicators for these producers were taken over from the sectoral accounts as units in question are grouped into sub-sector public non-financial institutions (S11001).

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4 For more detailed analysis see Vebrova, Rybacek (2014).

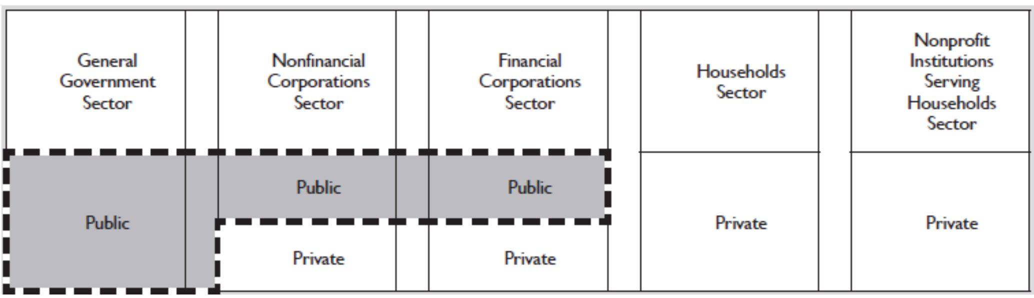
5 For this text, macroeconomic indicators as published by the CZSO at the end of January 2016 were used.

This holds true also for the central bank which represents another important component of the public sector. Units conducting monetary policy, which constitutes deliberate regulation of monetary conditions whereas final goals are defined by the law, are classified in the sub-sector S12001 which is separately published so that data could be easily taken over. However, other public financial institutions are not shown in one individual sequence of accounts so that appropriate data must be estimated using an alternative method.

This data gap has been filled by identification of the most significant units (public banks and public insurance companies), using their individual data and its integration into macroeconomics indicators of the public sector. These estimations make the picture of the public sector more complete, so the accounts for public institutions encompass government units, all public non-financial public corporations and a dominant part of public financial corporations.

The following box summarizes the structure of public sector as described above.

Box 1 The structure of the public sector



Source: Public Sector Debt Statistics: Guide for Compilers and Users (2013)

In the analysis below, the aggregates on total revenues and expenditures expressed as percentage of GDP will be applied which leads us to the consolidation issue. Consolidation of data assures that the public sector is shown as it were one unit. Generally, consolidated data are preferred to non-consolidated one (O´Connor, Weisman, Wickens, 2004). It is thus preferred to eliminate the internal flows. One of the main reasons is to have reliable tool how to assess overall impact of government operations on the rest of the economy.

However, the reasoning of the consolidation is mainly focused on the case of flows. When assessing the influence of the public sector on the economy, consolidated data on flows are strongly preferred to avoid double-counting. As O´Connor, Weisman and Wickens (2004) stated, it will prevent the distorting effects due to different institutional arrangements across countries. The larger is the number of existing government institutions the higher is the frequency of transactions between them, so total revenues and expenditures will inflate.

In case of stocks, the situation is very different. Even internal debt reflects the financial stability of units constituting the public sector. If a public unit provides a loan to another public unit, it indicates that one public unit (debtor) is in a need to receive additional financial resources whereas the other unit (creditor) now faces the counterparty risk. So, the financial risk in the public sector has clearly increased; newly formed financial relation poses a threat to financial soundness of public units concerned.

Furthermore, consider commercial private bank selling central government bonds to local government unit. If this is the case, government bonds disappeared from consolidated debt statistics as if they were repaid or cancelled by counterparty. However, they still exist; moreover, the financial situation of one government unit (creditor) depends, to some extent, on the ability of other government unit (debtor) to repay its debt. At the level of individual units, the "internal debt" simply matters.

We should briefly touch the double-counting issue. Even if it cannot be denied that double-counting of debt instruments is conceivable, it would be rather rare case when, e.g., a public unit takes a loan from non-public institutions and channels newly received financial means to another public unit again in form of loan. Due to rare occurrence in the Czech reality in the investigated period, obtaining more complete picture of the public sector indebtedness will clearly outweigh the risk of double-counting. Based on this reasoning, the approach in the following analysis assumes the use of consolidated data for flows, while for the assessment of financial stability non-consolidated data on liabilities are exploited.

Flows in the public sector (mainly between the general government and other parts of the sector) can take wide range of forms including taxes, subsidies, interests or transfers of assets. Consolidation of data is very demanding on the availability of relevant information on counterparties. However, in some cases it could be assumed that both parties are classified within the public sector so that given flows (respective national account´s item) can be excluded in its entirety. This is the case of taxes paid by public units, subsidies or capital transfers on resources side of public units. The items just mentioned were consolidated reflecting the recommendation to eliminate the most important flows (O´Connor, Weisman and Wickens, 2004).

As mentioned above, financial soundness of the public sector will be assessed on the base of non-consolidated data. This approach is justifiable as "internal" debt also poses potential threats to the financial sustainability of the public sector. At this stage, it is worth to clarify the differences between EDP debt used for monitoring of public finance in the EU. EDP debt, presented as consolidated indicator containing selected balance sheet items, covers only government units whereas the debt is valued at nominal value. On the contrary, the public debt is defined more broadly as total sum of liabilities valued preferably at market prices. Public debt also covers, by definition, all public units.

Before the analytical part, some specifics of the Czech economy should be mentioned. In the last decades, the Czech economy has gone through profound institutional changes and a number of significant flows in terms of their nominal amounts have been carried out. But such flows would blur the analysis of long-term trend in the public sector development. To eliminate this "noise", we have identified significant (mainly) expenditures or transfers[6] of assets whose impact on the nominal level of revenues and expenditures is prevented. Data obtained by this procedure are used for drawing an alternative picture on the dynamics of the public sector.

3 DEVELOPMENT OF THE PUBLIC SECTOR IN THE CZECH REPUBLIC

The role which the public sector plays in the economy can be expressed in different ways, or rather by a number of indicators. It is our opinion that proper analysis cannot be based on one indicator only as this is typically the case of value added. Relying on one or two chosen indicators is the same as to look only ahead before crossing the road; we could omit or lost very important information on what is going on in the economy. Thus, we use both flows and stocks to describe a share which is taken by the public sector in the Czech Economy.

The following table shows the development of output (the item P.1 in national accounts) and value added of the public sector and corresponding shares on total values. The share of output has fallen by four percentage points (p.p.) over the period under investigation; however, the share on total value added has declined only by 1.5 p.p. over this period. This can be generally explained by a decline in intermediate consumption exceeding the decline in output.

[6] This is the case of transformation expenditures (losses of transformation institutions, etc.), privatisation of assets below their market value (whereas the difference is classified as expenditure of government sector) or church restitutions.

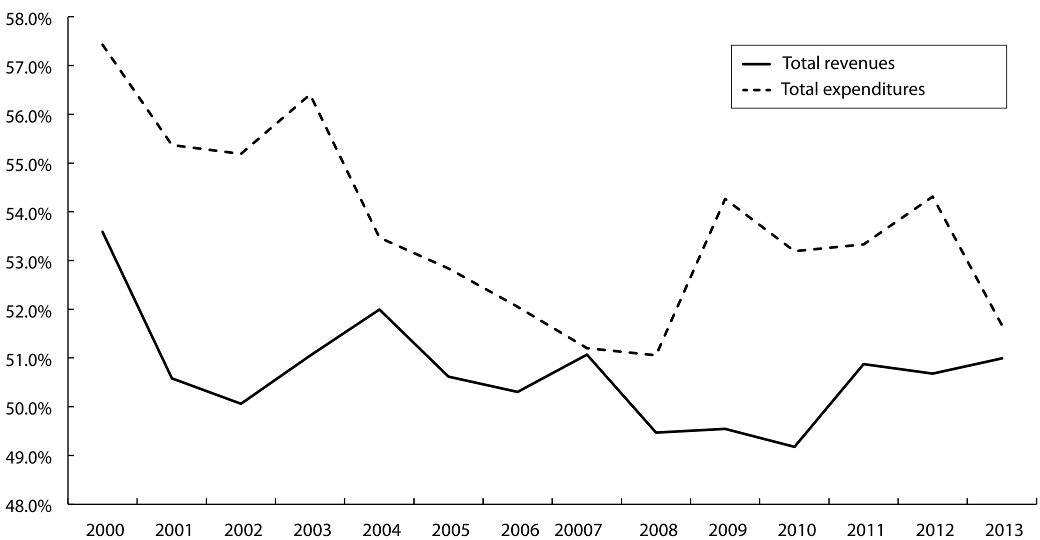
Table 1 Output and value added of the public sector, the Czech Republic (CZK mill., 2000–2013)

	2000	2001	2002	2003	2004	2005	2006
Output	895 544	865 940	893 618	970 618	1 028 461	1 037 498	1 104 494
– share in total output	16.9%	15.1%	15.0%	15.2%	14.6%	13.9%	13.2%
Value added (VA)	487 632	484 312	512 106	548 433	575 637	599 407	646 723
– share in total VA	20.8%	19.8%	19.9%	19.8%	19.5%	18.8%	18.6%
	2007	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**
Output	1 173 101	1 223 302	1 250 919	1 230 185	1 257 917	1 239 570	1 245 950
– share in total output	12.7%	12.7%	14.1%	13.3%	13.0%	12.9%	12.9%
Value added	682 030	724 809	754 590	737 607	736 878	736 071	743 234
– share in total VA	18.7%	20.4%	21.1%	20.3%	20.2%	20.1%	19.3%

Source: <www.czso.cz>, own calculation

In other words, productive activities of public sector expressed as total output has been declining while in terms of value added the range of activity remains more or less stable. Nevertheless these figures can only tell the story about the development of productive activities as defined in the methodology of national accounts. To make one step further, we take the information on total revenues and expenditures the public units as these are defined in national accounts. These aggregates supposedly better reflect the overall role of the public sector in society. The following chart graphs the development of total revenues and expenditures expressed as shares in GDP.

Figure 1 Total expenditures and total revenues of the public sector, consolidated (% of GDP, 2000–2013)

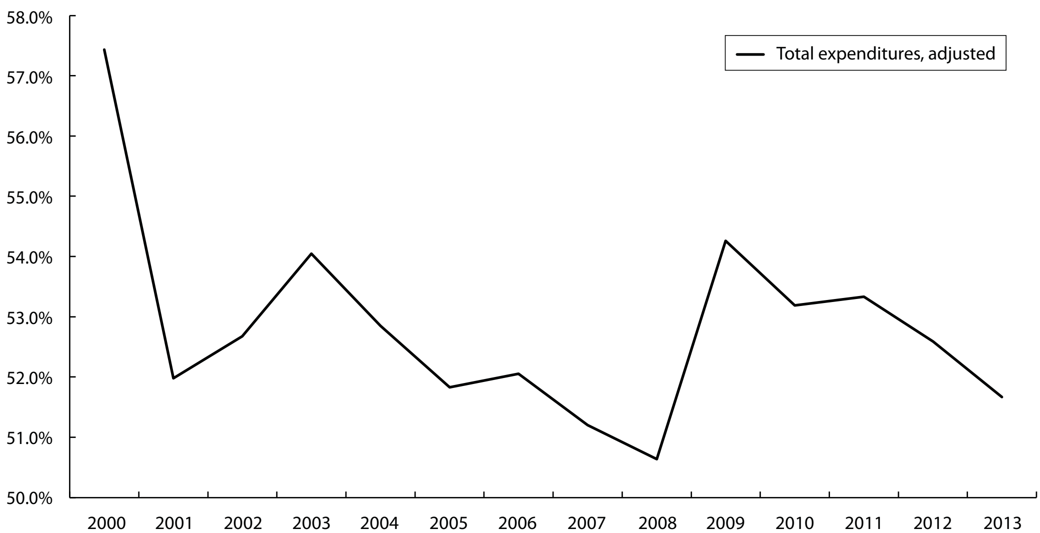


Source: <www.czso.cz>, own calculation

Both shares have been declining before the outset of the current recession starting in 2009. At that time, the trend of total expenditures has abruptly changed. Compared to the previous year, the share of total expenditures has shot up by more than 3 p.p. in 2009 while the share of total revenues has remained stable. It can be drawn from the chart that despite of both privatisation process and declining share in total value added, the role of the public sector currently tends to rise. This trend has become obvious especially since 2009.

As has been mentioned in the methodological part of the text, the analysis is supplemented with the aggregates adjusted by exceptional expenditures undertaken by government which are related to the transformation of the Czech economy or which otherwise bring a noise to the assessment whether the role of public sector tends to grow or not.[7] Adjusted time series is presented in the following chart. The results seem to be very similar to the previous analysis. However, from the long-term perspective, the share of total expenditures of the public sector in total GDP remains more stable with a tendency to grow since 2009 when the recession hits the economy.

Figure 2 Total adjusted expenditures of the public sector, consolidated (% of GDP, 2000–2013)



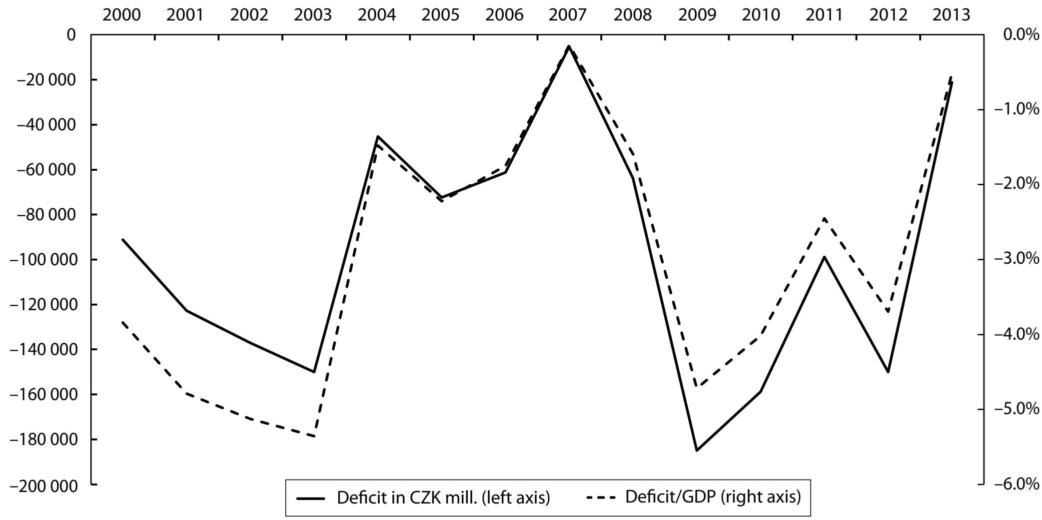
Source: <www.czso.cz>, own calculation

So from the perspective of flows, the public sector has been showing a declining share of productive activities in terms of output or value added. This trend is accompanied by rather stable or even rising trend in last years as indicated by the share of total revenues and expenditures of the public sector in GDP. The figures thus indicate that the public sector has become less oriented on productive activities along with stronger orientation on redistribution of funds circulating in the economy.

When studying the behaviour of any sector, the indicator of deficit/surplus offers very reliable input into the assessment of the financial situation. As can be drawn from the following chart, the public sector in the Czech Republic generates permanently quite huge deficits so that balanced budget has never been achieved during the investigated period. It is worth mentioning that this indicator is not influenced by the fact whether the data is consolidated or not.

[7] See footnote 4.

Figure 3 The public sector deficit, CZK mill. (% of GDP, 2000–2013)



Source: <www.czso.cz>, own calculation

Now we turn to the balance sheet items where public sector acts as owner of non-financial and financial assets and as debtor. The balance sheet items provide very important information mainly because they indicate the way of funding. As can be read from the following table, the public sector owns more than 50 percent of non-financial assets, but this share has fallen by 9 p.p. throughout this period. This decline can be explained by continuing privatisation process (housing, church restitutions, etc.) or by lower investments of public sector into non-financial assets.

We can conclude that there is a downward trend in the share of the public sector as far as non-financial assets are concerned. On the liability side, the situation is quite different. The public sector takes larger part of total value of liabilities than in 2000; the share has gone up by more than 2 percentage points. This

Table 2 Non-financial assets and liabilities of the public sector (CZK mill., 2000–2013)

	2000	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**
Non-financial assets (NFA)	13 153	13 187	13 168	13 080	13 186	13 670	13 910
– share in total NFA	63.0%	61.3%	60.4%	59.2%	58.1%	57.7%	56.3%
Liabilities	2 178	2 246	2 809	2 881	3 166	3 463	3 553
– share in total liabilities	22.3%	22.4%	25.6%	24.0%	25.0%	25.3%	24.6%
	2007	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**
Non-financial assets (NFA)	14 499	15 058	15 183	15 187	15 539	15 605	15 162
– share in total NFA	54.7%	53.7%	54.0%	53.5%	53.3%	53.2%	52.1%
Liabilities	3 841	3 961	4 278	4 526	4 644	4 991	5 045
– share in total liabilities	23.7%	23.5%	24.3%	25.0%	24.7%	25.7%	24.5%

Source: <www.czso.cz>, own calculation

can be rather surprising when we consider the trends in other indicators analysed above. It can confirm one of the aspects of the theory explained by Buchanan and Wagner (2000) which suggests that public sector has inherent tendency to intensify debt financing when providing services to the society.

When going on the level of individual items, we find out that mainly issuance of bonds in the public sector has been taken larger part of financing. Relevant comparison is exemplified in the following table.

Table 3 Changes on the liability side from 2000 to 2013 (in %)

Changes from 2000 to 2013	Total liabilities	Debt securities
Domestic economy (net of public sector)	205%	216%
Public non-financial corporation	122%	595%
General government sector	432%	672%

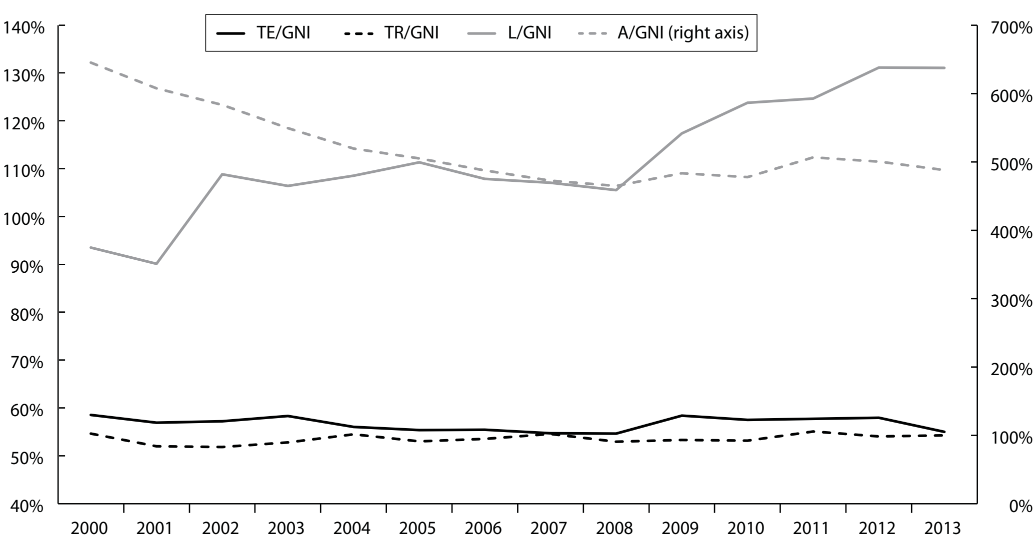
Source: <www.czso.cz>, own calculation

In the non-public part of the economy, the amount of total liabilities has more than doubled whereas the same holds true for debt securities. In case of public non-financial corporation, total liabilities have risen moderately, but debt issuance has strongly exceeded the trend in the non-public economy. It becomes more evident in the case of general government sector. Total liabilities have quadrupled and the change in the amount of debt securities issued by government sector has gone up sevenfold.

Even if we do not consider the effect of revaluation, it can be concluded that public sector tends very strongly to debt funding which is getting increasingly used for the financing of redistribution rather than production. Buchanan and Wagner (2000) concluded that inherent tendency of public financing is increasing size of public sector expressed as its share in gross national income (GNI). The following chart shows the development of public sector aggregates in the period under investigation.

From the long-term perspective, the level of total expenditures (TE) and total revenues (TR) of the public sector remains more or less stable ranging from 52 to 59 percent. The share of total assets owned by public sector has declined by 25 percentage points whereas this trend is influenced also by

Figure 4 The public sector deficit, CZK mill. (% of GDP, 2000–2013)



Source: <www.czso.cz>, own calculation

privatisation process. The most evident course is visible on the liability side of the public sector; the share of total liabilities of the public sector in GNI has grown by more than 40 percent.

Depending on the way the size of the public sector is expressed, we can make number of conclusions on the course of public sector´s size. One of the main points of Buchanan and Wagner is that due to *inter alia* increasing share of debt funding, decreasing perceived prices of public goods and services lead to the shift of scarce resources and financial means toward the public sector. Thus, as Buchanan and Wagner (2000) deal primarily with financial behaviour of the public sector, this analysis tends to confirm, in the last period, that there is a growing trend in the size of the public sector in terms of its expenditures and especially the amount of liabilities.

CONCLUSION

The text is focused on the public sector statistics which can be very manifold in analytical use. The satellite accounts of public sector represent useful tool overcoming the ambiguities of core national accounts as it groups together units through which government policy is implemented. It can supposedly better reflect the overall role of government in the society. We have used most recent data published by the Czech Statistical Office whereas the time series has been prolonged enabling to analyse long-term performance of the public sector in the Czech Republic. As public sector´s liabilities showed a strong rise exceeding that in the rest of the economy, the analysis confirms the conclusions of Buchanan and Wagner (2000) on the tendency of deficit and debt financing of public goods provision as the share of total liabilities of the public sector in total liabilities in the economy has grown by 40 percentage points expressed as share in GNI. Moreover, the final balance of the public sector revenues and expenditures indeed ends up in deficit irrespective of the phase of business cycle. On the other side, the share of total revenues and expenditures remains rather stable over time slightly above 50 percent of GDP which is not supportive of the theory suggesting the inherent tendency of the public sector to expand. Declining trend is evident in the case of total assets, the public sector ownership has declined by 9 percentage points expressed as share of assets held by the public sectors on total value of assets in the economy; nevertheless, these trends are undoubtedly affected by privatization process which has been taking place since the nineties.

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Reduction in CPI Commodity Substitution Bias by Using the Modified Lloyd–Moulton Index

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Abstract

The Consumer Price Index (CPI) is used as a basic measure of inflation. In practice, the Laspeyres price index is used to measure the CPI, although this formula does not take into account changes in the structure of consumption. The difference between the Laspeyres index and the superlative index should approximate the value of the commodity substitution bias. The Lloyd–Moulton price index does not make use of current-period expenditure data and, as it is commonly known, it allows to approximate superlative indices, in particular the Fisher price index (Von der Lippe, 2007). This is a very important property for the inflation measurement and the Consumer Price Index bias calculations. In this paper we verify the utility of the Lloyd–Moulton price index as the Fisher price index approximation. We propose a simple modification of that index and verify this modification for the real data set.

Keywords	***JEL code***
Price indices, Fisher index, Lloyd–Moulton index, Consumer Price Index	*C43*

INTRODUCTION

The Consumer Price Index (CPI) is used as a basic measure of inflation. In practice, the Laspeyres price index is used to measure the CPI, although this formula does not take into account changes in the structure of consumption. The difference between the Laspeyres index and the superlative index should approximate the value of the commodity substitution bias. The Lloyd–Moulton price index does not make use of current-period expenditure data and, as it is commonly known, it allows to approximate superlative indices, in particular the Fisher price index (Von der Lippe, 2007). This is a very important property for the inflation measurement and the Consumer Price Index bias calculations. In this paper we verify the utility of the Lloyd–Moulton price index as the Fisher price index approximation. We propose a simple modification of that index and verify this modification for the real data set.

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INTRODUCTION

The Consumer Price Index (CPI) is used as a basic measure of inflation. The index approximates changes in the costs of household consumption assuming the constant utility (COLI, Cost of Living Index). In practice, the Laspeyres price index is used to measure the CPI (see White, 1999; Clements and Izan, 1987) although this formula does not take into account changes in the structure of consumption. It means that the Laspeyres index can be biased due to the commodity substitution because relative prices change over time and also consumers' preferences. Many economists consider the *superlative indices* (see Von der Lippe, 2007) to be the best approximation of COLI. The difference between the Laspeyres index and the superlative index should approximate the value of the commodity substitution bias. However there is a way to reduce that bias: using a Constant Elasticity of Substitution (CES) framework, a superlative price index can be approximated once we have estimated the elasticity of substitution. The Lloyd–Moulton price index (see Lloyd, 1975; Moutlon, 1996; Shapiro and Wilcox, 1997) does not make use of current-period expenditure data, so it is even possible to approximate a superlative index (like the *ideal* Fisher index) in real time and extrapolate the time series. In this paper we verify the applicability of the Lloyd–Moulton price index as the Fisher price index approximation. We propose a modification of the Lloyd–Moulton price index which facilities numerical calculations. We also examine the modified Lloyd–Moulton price index on the real data set for Poland.

1 SUPERLATIVE PRICE INDICES IN CPI BIAS MEASUREMENT

An interesting discussion on the theory of the COLI can be found in the following papers: Diewert (1993), Jorgenson and Slesnick (1983), Pollak (1989). Let $E(P,\bar{u}) = \min_Q\{P^T Q | U(Q) \geq \bar{u}$ be the expenditure function of a representative consumer which is dual to the utility function $U(Q)$. In other words it is the minimum expenditure necessary to achieve a reference level of utility $\bar{u}$ at vector of prices P. Then the Konüs cost of the living price index is defined as (see Von der Lippe, 2007):

$$P_K = \frac{E(P^t,\bar{u})}{E(P^s,\bar{u})}, \tag{1}$$

where t denotes the current period, s denotes the base period, and in general, the vector of N considered prices at any moment τ is given by $P^\tau = [p_1^\tau, p_2^\tau, ..., p_N^\tau]'$. P_K is the true cost of living index in which the commodity Q changes together with the vector of prices facing the consumer changes. The CPI, in contrast, measures the change in the cost of purchasing a fixed basket of goods over the time interval, i.e. $Q^s = [q_1^s, q_2^s, ..., q_N^s]' = Q^t$. The CPI is a Laspeyres-type index defined by:

$$P_{La} = \frac{\sum_{i=1}^{N} q_i^s p_i^t}{\sum_{i=1}^{N} q_i^s p_i^s}, \tag{2}$$

so we assume here the constant consumption vector on the base period level. It can be shown (see Diewert, 1993) that under the assumption that the consumption vector Q^t solves the period t expenditure minimization problem, then

$$P_K = \frac{E(P^t,U(Q^s))}{E(P^s,U(Q^s))} \leq P_{La}, \tag{3}$$

and thus $P_{La} - P_K$ is the extent of the commodity substitution bias, where P_K plays the role of the reference benchmark. In the so called economic price index approach the superlative price indices are treated as the best approximation of the P_K index (see White, 1999).

We define a price index P to be *exact* for a linearly homogeneous aggregator function f (here a utility function), which has a dual unit cost function $c(\cdot)$ and it holds

$$P = \frac{c(P^t)}{c(P^s)}. \tag{4}$$

In other words, an *exact* price index is the one whose functional form is *exactly* equal to the ratio of cost functions for some underlying functional form representing preferences. The Fisher price index P_F is exact for the linearly homogeneous quadratic aggregator function $f(x) = (x^T Ax)^{0.5}$, where A is a symmetric and positive matrix of constants (Diewert, 1976). The quadratic function above is an example of a *flexible functional form* (i.e. a function that provides a second order approximation to an arbitrary twice continuously differentiable function). Since P_F is exact for a flexible functional form, it is said to be a *superlative* index number. In Afriat (1972), Pollak (1971) and Samuelson-Swamy (1974) we can find other examples of exact index numbers as well as superlative index numbers. The Fisher price index is defined as a geometric mean of the Laspeyres and Paasche indices (P_{Pa}), where:

$$P_{Pa} = \frac{\sum_{i=1}^{N} q_i^t p_i^t}{\sum_{i=1}^{N} q_i^t p_i^s}. \tag{5}$$

Let us notice that the Fisher price index makes use of current-period expenditure data and its usefulness in CPI measurement is limited.

2 LLOYD–MOULTON PRICE INDEX

The quadratic mean of order r price index was defined by Diewert (1976) as follows ($r \neq 0$):

$$P_{QM}(r) = \{[\sum_{i=1}^{N} w_i^s (\frac{p_i^t}{p_i^s})^{\frac{r}{2}}]^{\frac{2}{r}} [\sum_{i=1}^{N} w_i^t (\frac{p_i^t}{p_i^s})^{-\frac{r}{2}}]^{-\frac{2}{r}}\}^{\frac{1}{2}}, \tag{6}$$

where w_i^s and w_i^t denote the expenditure share of commodity i in the base period s and the current period t, respectively. It is a superlative price index. By setting $r = 2(1 - \sigma)$ expression (6) becomes:

$$P_{QM}(2(1-\sigma) = \sqrt{P_{LM}(\sigma) \cdot P_{CW}(\sigma)}, \tag{7}$$

where $P_{LM}(\sigma)$ denotes the Lloyd-Moutlon price index defined as:

$$P_{LM}(\sigma) = \{[\sum_{i=1}^{N} w_i^s (\frac{p_i^t}{p_i^s})^{1-\sigma}]^{\frac{1}{1-\sigma}}, \tag{8}$$

and $P_{CW}(\sigma)$ denotes its "current weight (CW) counterpart" (see De Haan et al., 2000), i.e.:

$$P_{CW}(\sigma) = \{[\sum_{i=1}^{N} w_i^t (\frac{p_i^t}{p_i^s})^{-(1-\sigma)}]^{\frac{-1}{1-\sigma}}. \tag{9}$$

These formulas $P_{LM}(\sigma)$, $P_{CW}(\sigma)$ and $P_{QM}(\sigma)$ do not exists for, $\sigma = 1$, but for $\sigma \rightarrow 1$ these indices tend to Geometric Laspeyres, Geometric Paasche and Törnqvist price indices, respectively (De Haan et al., 2000). Since the price index $P_{LM}(\sigma)$ monotonically decreases and $P_{CW}(\sigma)$ monotonically increases as σ increases (see Biggeri, Ferrari, 2010) we conclude that there exists a value σ_0 which satisfies:

$$P_{LM}(\sigma_0) = P_{CW}(\sigma_0) = P_{QM}(2(1-\sigma_0)). \tag{10}$$

Thus we get that for $\sigma = \sigma_0$ the Lloyd–Moulton index becomes superlative. The value σ_0 should be obtained numerically. Since the superlative Fisher price index satisfies the most important tests from the axiomatic price index theory (Fisher, 1922; Balk, 1995) the approximation $P_{LM}(\sigma_0) \approx P_F$ is desirable. In other words we are going to find such value σ_0 that minimizes the expression $|P_{LM}(\sigma) - P_F|$.

Remark

The numerical methods need some starting assumptions about the interval of possible values of the given parameter (or parameters). If the interval is wide the methods could be computationally inefficient. It is not convenient if we must establish the interval for numerical calculations each time we change the starting set of random variables. It would be ideal to have a fixed, narrow interval including the value of the parameter σ_0. In our opinion we can not recommend one general value of the parameter σ or even the interval of its possible values (see Example) although some papers suggest it is a number between 0.7 and 1 (Shapiro, Wilcox, 1997; Biggeri, Ferrari, 2010). The value of the elasticity (σ) depends on the aggregation level: at a detailed levels of product aggregation the substitution elasticity could well be above 1 (see e.g. Balk, 2000). In this paper we propose some simple modification of the Lloyd–Moulton price index which makes that the estimated value of the parameter (ϕ) is always in the interval (0, 1) and thus ϕ should not be treated as the elasticity of substitution.

Let us replace $1 - \sigma$ by $\dfrac{1}{ctg(\frac{\pi\phi}{2})}$ in expression (8) for the Lloyd-Moulton index.

We obtain:

$$P_{LM}(\phi) = \{[\sum_{i=1}^{N} w_i^s (\frac{p_i^t}{p_i^s})^{\frac{1}{ctg(\frac{\pi\phi}{2})}}]^{ctg(\frac{\pi\phi}{2})}. \tag{11}$$

Since the function $ctg(\pi\phi/2)$ is decreasing with respect to $\phi \in (0,1)$, $ctg(0^+) = \infty$ and $ctg(\frac{\pi}{2}) = 0$ we can consider only $\phi \in (0,1)$. In fact we have:

$$\lim_{\phi \to 1^-} P_{LM}(\phi) = \lim_{x \to 0^+} (\sum_{i=1}^{N} w_i^s (\frac{p_i^t}{p_i^s})^{\frac{1}{x}})^x = \lim_{x \to 0^+} \exp(x \ln(\sum_{i=1}^{N} w_i^s (\frac{p_i^t}{p_i^s})^{\frac{1}{x}})) \le$$

$$\le \lim_{x \to 0^+} \exp(x \ln(\max_i \frac{p_i^t}{p_i^s})^{\frac{1}{x}}) = \exp(\ln(\max_i \frac{p_i^t}{p_i^s})) = \max_i \frac{p_i^t}{p_i^s}. \tag{12}$$

We are going to find such value ϕ_0 that minimizes the expression $|P_{LM}(\phi) - P_F|$. Finding the minimum of $|P_{LM}(\phi) - P_F|$ in Mathematica 6.0 the best estimated solution, with feasibility residual, Karush–Kuhn–Tucker (KKT) residual or complementary residual, is returned.

Example

Let us consider the case when vectors of prices and quantities of $N = 12$ commodities are described as follows:

$P^t = [a \cdot 700, a \cdot 1900, 400, 8, 120, 120, 1200, 1000, a \cdot 500, 5, 120, 2200]'$;

$P^s = [800, 1700, 300, 9, 130, 1300, 900, 1700, 560, 6, 135, 2300]'$;

$Q^t = [400, 200, 5000, 500, 340, 700, 800, 500, 3000, 500, 340, 700]'$;

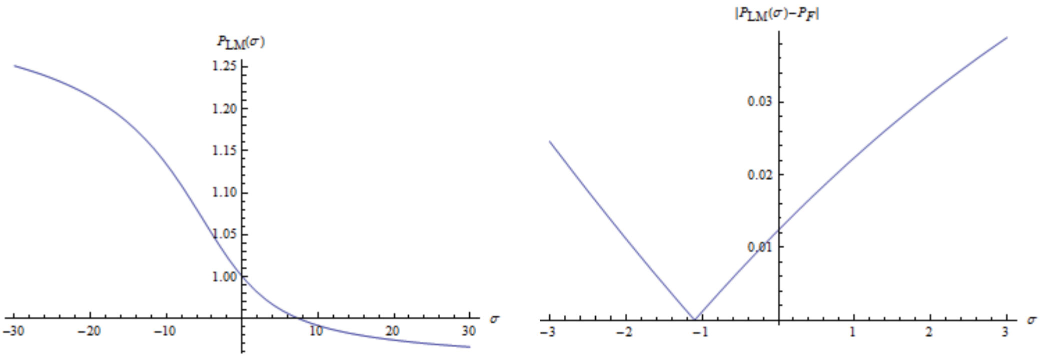
$Q^s = [350, 350, 4000, 800, 450, 700, 550, 400, 5000, 700, 250, 700]'$;

where a is some positive parameter which influences on the sign of a difference $P_{La} - P_F$.

After calculations for $a = 1$ we obtain $P_{La} = 1.000$, $P_{Pa} = 1.025$ and $P_F = 1.012$. Functions $P_{LM}(\sigma)$ and $|P_{LM}(\sigma) - P_F|$ depending on σ (still $a = 1$) are presented in Figure 1. In the same case functions $P_{LM}(\phi)$ and $|P_{LM}(\phi) - P_F|$ depending on ϕ are presented in Figure 2.

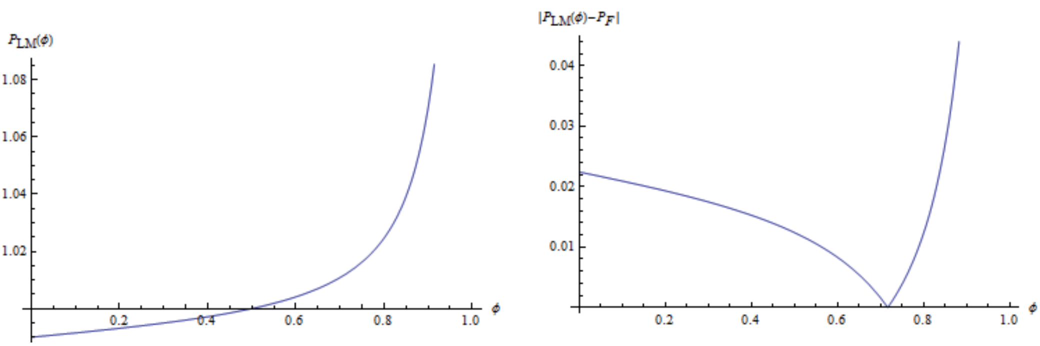
Values σ_0, ϕ_0, P_{La}, P_F, $|P_{LM}(\sigma_0) - P_F|$ and $|P_{LM}(\phi_0) - P_F|$ for different values of a are presented in Table 1. We observe $\phi \in (0,1)$ since σ_0 can be positive or negative.

Figure 1 Values $P_{LM}(\sigma)$ and $|P_{LM}(\sigma) - P_F|$ depending on σ



Source: Mathematica 6.0

Figure 2 Values $P_{LM}(\phi)$ and $|P_{LM}(\phi) - P_F|$ depending on ϕ



Source: Mathematica 6.0

Table 1 Values σ_0, ϕ_0, P_{La}, P_F, $|P_{LM}(\sigma_0) - P_F|$ and $|P_{LM}(\phi_0) - P_F|$ for different values of a

Parametr a	a = 0.8	a = 0.9	a = 1	a = 1.1	a = 1.2	a = 1.3
P_{La}	0.921	0.960	1.000	1.039	1.078	1.118
P_F	0.946	0.979	1.012	1.0456	1.078	1.111
σ_0	−1.106	−1.192	−1.098	−0.674	0	0.529
$\lvert P_{LM}(\sigma_0) - P_F\rvert$	$2.633 . 10^{-6}$	$1.575 . 10^{-7}$	$3.6 . 10^{-8}$	$1.317 . 10^{-8}$	$6.677 . 10^{-5}$	$1.307 . 10^{-8}$
ϕ_0	0.703	0.727	0.716	0.657	0.502	0.280
$\lvert P_{LM}(\phi_0) - P_F\rvert$	0.0002	$3.965 . 10^{-7}$	$2.584 . 10^{-7}$	$1.511 . 10^{-7}$	$8.311 . 10^{-8}$	$5.927 . 10^{-8}$

Source: Mathematica 6.0

3 EMPIRICAL STUDY

In our empirical illustration of the presented method of the CPI bias reduction we use monthly data[2] on price indices of consumer goods and services in Poland for the time period Jan. 2010–Jan. 2013 (36 observations). The weights w_i^s and $w_{i,t}$ also are taken from data published by the Central Statistical Office.[3] Very low CPI commodity substitution bias was observed in Poland in the period under study (the largest for the data from the period of Jan. 2010–Jan. 2011, less than 0.034 percentage points, and the smallest for the data from the period of Jan. 2011–Jan. 2012, 0.013 percentage points – see Białek, 2014). This is in part due to the frequent annual, update of the weights in the CPI basket of goods in Poland. The results described in the study indicate that there is virtually no difference whether this bias is measured with the Fisher superlative index or the Törnqvist superlative index. In the period under study, although the CPI bias should be considered as small, it is positive for each year (relative to the Laspeyres index). This conclusion corresponds to the results of most studies in the world – similar results were observed in Germany (Hoffmann, 1999), Sweden (Dahlen, 1994), the Czech Republic (Filer, Hanousek, 2003), and Australia (Woolford, 1994). However, in some countries the CPI commodity substitution bias proved many times larger (e.g., in the US – see Boskin et al., 1996). Moreover, the size of the CPI substitution bias may be bigger if the system of weights is updated rarely. This is just the reason of our study where we intend to verify the scale of a reduction in CPI substitution bias by using the modified Lloyd–Moulton index (see formula 11) and under the consideration not only the frequent annual, update of the weights in the CPI basket of goods in Poland.

Let us notice that having expenditure shares of commodity i in the base period s and the current period t we can express the Laspeyres and Paasche formulas as follows:

$$P_{La} = \sum_{k=1}^{N} w_k^s P_k^{s,t}, \tag{13}$$

$$P_{Pa} = \frac{1}{\sum_{k=1}^{N} \frac{w_k^t}{P_k^{s,t}}}, \tag{14}$$

where $P_k^{s,t}$ denotes the k – th price relative (partial index) for the compared time moments s and t, and it is obviously published by the Central Statistical Office. The first step of the study is to compare the reduced CPI substitution bias (i.e. $|P_{LM}(\phi_0) - P_F|$) for each yearly period of time, i.e. a) Jan. 2010–Jan. 2011 b) Jan. 2011–Jan. 2012 and c) Jan. 2012–Jan. 2013. Our results are presented in Table 2 and in Figure 3.

[2] We use highly-aggregated data taking into account price indices of the following group of consumer goods and services in Poland: food and non-alkoholic beverages (*X1*), alcoholic beverages, tobacco (*X2*), clothing and footwear (*X3*), housing, water, electricity, gas and other fuels (*X4*), furnishings, household equipment and routine maintenance of the house (*X5*), health (*X6*), transport (*X7*), communications (*X8*), recreation and culture (*X9*), education (*X10*), restaurants and hotels (*X11*) and miscellaneous goods and services (*X12*). The author is aware of the fact that the presented calculations play only a role of some illustration. Drawing any serious conclusions must be based on data from a lower level of aggregation, preferably at the 4-digit class level of the COICOP (Classification of Individuals Consumption according to Purpose).

[3] Główny Urząd Statystyczny (GUS) in Poland.

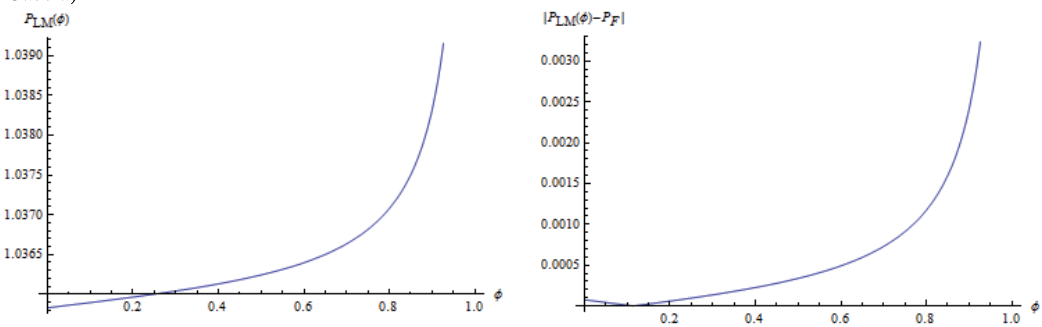
Table 2 Values P_{La}, P_F, ϕ_0 and $|P_{LM}(\phi_0) - P_F|$ for different time periods

Time period	Jan. 2010–Jan. 2011	Jan. 2011–Jan. 2012	Jan. 2012–Jan. 2013
P_{La}	1.0362	1.0389	1.0169
P_F	1.0359	1.0397	1.0167
ϕ_0	0.1142	0.6075	0.0756
$\lvert P_{LM}(\phi_0) - P_F\rvert$	$1.89 \cdot 10^{-9}$	$5.61 \cdot 10^{-9}$	$1.002 \cdot 10^{-9}$

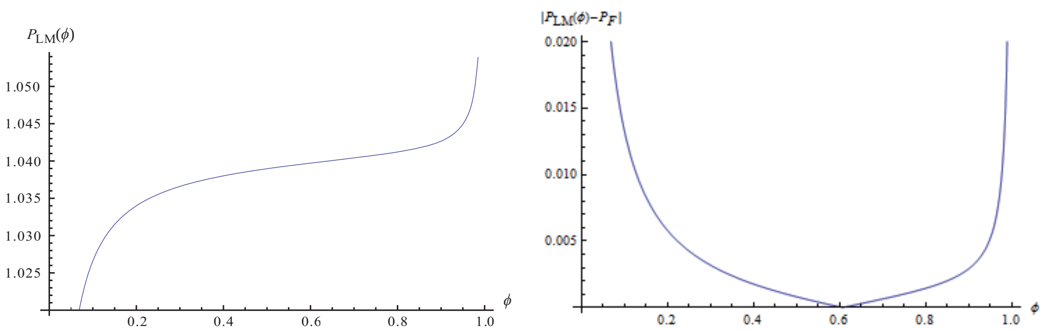
Source: Mathematica 6.0

Figure 3 Values $P_{LM}(\phi)$ and $|P_{LM}(\phi) - P_F|$ depending on ϕ for different time periods

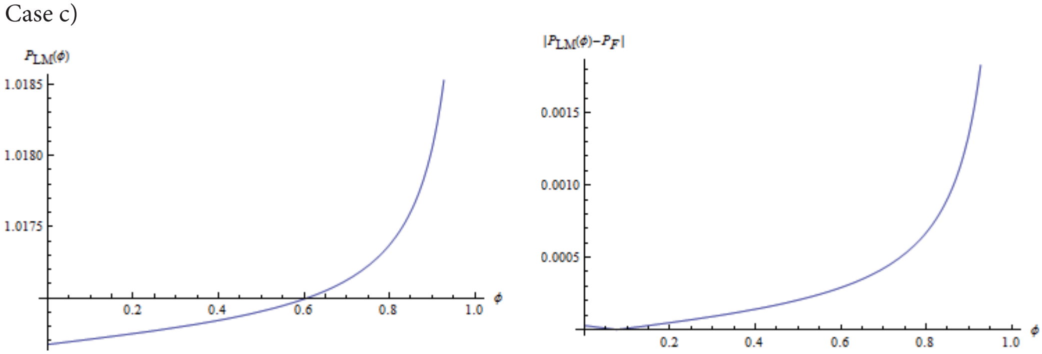
Case a)



Case b)



Case c)



Source: Mathematica 6.0

The next step of the study is verifying the hypothesis that we can eliminate the CPI substitution bias using the Lloyd–Moulton price index even if we do not update the weights in the CPI basket of goods each year. Only for the Lloyd–Moulton price index calculations we assume that weights in the CPI basket of goods are from Jan. 2010 (outdated) and we consider the CPI substitution bias for time periods: Jan. 2011–Jan. 2012 and Jan. 2012–Jan. 2013. The results are presented in Table 3 and in Figure 4.

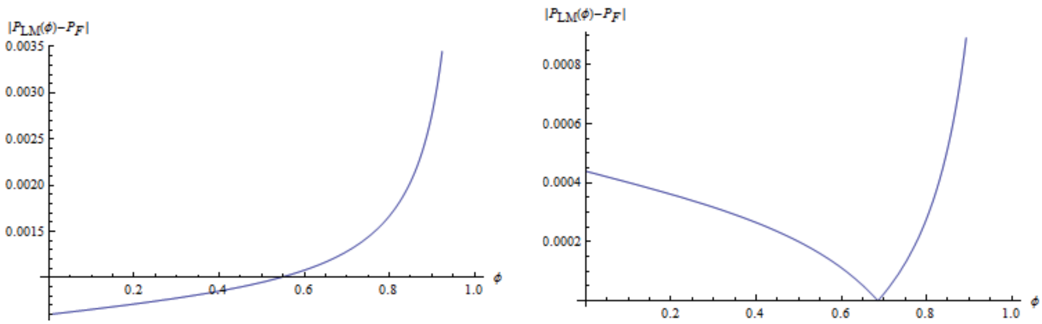
Table 3 Values P_{La}, P_F, ϕ_0 and $|P_{LM}(\phi_0) - P_F|$ for different time periods

Time period	Jan. 2011–Jan. 2012	Jan. 2012–Jan. 2013
P_F	1.0397	1.0167
ϕ_0	0.550	0.685
$\lvert P_{LM}(\phi_0) - P_F\rvert$	$6 \cdot 10^{-8}$	$4.6 \cdot 10^{-9}$

Source: Mathematica 6.0

Figure 4 Values $|P_{LM}(\phi) - P_F|$ depending on ϕ for periods: a) Jan. 2011–Jan. 2012 and b) Jan. 2012–Jan. 2013

Case a) Case b)



Source: Mathematica 6.0

CONCLUSIONS

The major advantage of using the modified Lloyd–Moulton price index (11) is that the value of the estimated parameter ϕ_0 is always in the interval (0,1) and calculations are faster. As we can notice (see Table 1) this rule is not satisfied in the case of the parameter σ_0 from the original Lloyd–Moulton index defined in formula 8. The empirical study shows that using the modified Lloyd–Moulton price index we can approximate the superlative Fisher index with an excellent precision. Thus, if we used the modified (or original) Lloyd–Moulton index instead of the Laspeyres price index in CPI calculations we would almost eliminate the CPI commodity substitution bias. The empirical study shows additionally that we can strongly reduce the CPI substitution bias using the modified Lloyd–Moulton price index even if we do not update the weights in the CPI basket of goods each year (see Table 3).

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Segmented Regression Based on Cut-off Polynomials

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Abstract

In *Statistika: Statistics and Economy Journal* No. 4/2015 (pp. 39–58), author's paper *Segmented Regression Based on B-splines with Solved Examples* was published. Use of B-spline basis functions has many advantages, the most important being a special form of matrix of system of normal equations suitable for quick solution of this system.

The subject of this paper is to explain how that segmented regression can be mathematically developed in other way, which doesn't require the knowledge of relatively complicated theory of B-spline basis functions, but is based on simpler apparatus of cut-off polynomials. The author considers a detailed calculation of matrix of system of normal equations elements and elaboration of so called polygonal method, as his contribution to issues of segmented regression. This method can be used to automatically obtain required values of nodal points. Author pays major attention to computing elements of matrix of system of normal equations, which he also developed as computer program called TRIO.

Keywords	***JEL code***
Normal equations, polygonal method, cut-off polynomials, linear, quadratic and cubic segmented regression, transformation of the parametric variable	*C10, C63, C65*

INTRODUCTION

This work deals with segmented regression based on splines as cut-off polynomials in three particular cases, so-called linear, quadratic, and cubic (segmented) regression. We introduce the so-called polynomial method of parametric-variable value assignment to experimentally obtained points (which lie generally in $\mathbb{R}^m$ of integer dimension $m \geq 1$), augmented with the computation of the so-called knot values of the variable corresponding to the division of these points into groups (segments).

To improve the numerical stability of parametric equations of the regression curves, we describe a transformation of the initial interval into a unit interval. Lastly, we choose the optimal regression curve for a given problem according to the measure of the determination indices of the three regression cases mentioned above.

Segmented regression can also be based on so-called B-spline functions. Through this method the matrix of the system of normal equations tridiagonal (for linear regression), five-diagonal (for quadratic regression), or seven-diagonal (for cubic regression) that enables a less elaborate solution to the given system, can be applied, see Kaňka (2015).

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1 NORMAL EQUATIONS

Let $n \geq 2$ be an integer, in the Euclidean space $\mathbb{R}^m$ ($m > 1$ integer) let us consider n points $P_i = x_j^{(i)}$ ($i = 1, ..., n; j = 1, ..., m$), not necessarily distinct (except for the case when all would be equal). Besides these points, for which we assume that $x_j^{(i)}$ are real random variables depending on a real variable t e.g. the time, we consider the so-called knots $T_0 < T_1 < ... < T_k$, where $k \geq 1$ is an integer, the so-called complementary knots $T_0 < T_1$ and $T_{k+1} > T_k$. The points $T_1 < T_2 < ... < T_k$ are called main knots.

In the intervals (T_{l-1}, T_l), $l = 1, ..., k + 1$, where the variable t changes, let us consider the increasing sequence $t_{l,1} < t_{l,2} < ... < t_{l,n(l)}, n(l) \geq 1$ integer, and let to every such member correspond exactly one point $x_j^{(lw)}$, $w = 1, ..., n(l)$. It holds then that $n = \sum_{l=1}^{k+1} n(l)$. The knots form the boundaries of the intervals, in the union of which we shall consider, depending on the chosen number $Q \in \{1, 2, 3\}$, the following real-valued functions of the real variable t

$$g_j(t) = \gamma_j^{(1)} + \gamma_j^{(2)}t + \cdots + \gamma_j^{(Q+1)}t^Q + \sum_{r=1}^{k} \gamma_j^{(r+Q+1)}[(t - T_r)_+]^Q, \tag{1}$$

where $\gamma_j^{(1)}, \gamma_j^{(2)}, ..., \gamma_j^{(k+Q+1)}$ are real parameters, i.e., linear (for $Q = 1$), quadratic (for $Q = 2$), and cubic (for $Q = 3$) splines in the form of so-called cut-off polynomials. By the symbol $(x)_+$ we will understand the following real-valued function of a real variable:

$$(x)_+ = \begin{cases} x, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

Instead of $[(t - T_r)_+]^Q$, we shall write in short $(t - T_r)_+^Q$.

Example 1.1

Let $k = 2, Q = 3, T_0 < T_1 < T_2 < T_3$, where $T_0 = 0, T_1 = 1, T_2 = 3, T_3 = 5$. There is (see formula 1):

$$g_j(t) = \gamma_j^{(1)} + \gamma_j^{(2)}t + \gamma_j^{(3)}t^2 + \gamma_j^{(4)}t^3 + \gamma_j^{(5)}(t - 1)_+^3 + \gamma_j^{(6)}(t - 3)_+^3,$$

thus (if we denote $\gamma_j^{(1)} + \gamma_j^{(2)}t + \gamma_j^{(3)}t^2 + \gamma_j^{(4)}t^3 = h_j(t)$)

$$g_j(t) = \begin{cases} h_j(t) & \text{for } 0 \leq t < 1, \\ h_j(t) + \gamma_j^{(5)}(t - 1)^3 & \text{for } 1 \leq t < 3, \\ h_j(t) + \gamma_j^{(5)}(t - 1)^3 + \gamma_j^{(6)}(t - 3)^3 & \text{for } 3 \leq t \leq 5. \end{cases}$$

We shall assume that the observed process is additive, i.e., for every possible value j, l, w the following holds:

$$x_j^{(lw)} = g_j(t_{lw}) + \epsilon_j^{(lw)},$$

where $\epsilon_j^{(lw)}$ are identically distributed random variables with the constant variance. Then we may obtain the estimates $c_j^{(1)}, c_j^{(2)}, \dots, c_j^{(k+Q+1)}$ of the parameters $\gamma_j^{(1)}, \gamma_j^{(2)}, \dots, \gamma_j^{(k+Q+1)}$ by the least squares method:

$$U_j = \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} [x_j^{(lw)} - g_j(t_{lw})]^2 =$$

$$= \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} \left[-x_j^{(lw)} + \sum_{q=1}^{Q+1} \gamma_j^{(q)} t_{lw}^{q-1} + \sum_{r=1}^{k} \gamma_j^{(r+Q+1)} (t_{lw} - T_r)_+^Q \right]^2.$$

By partial derivation we get for $1 \leq p \leq Q+1$ that:

$$\frac{\partial U_j}{\partial \gamma_j^{(p)}} = 2\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}\left[-x_j^{(lw)} + \sum_{q=1}^{Q+1}\gamma_j^{(q)}t_{lw}^{q-1} + \sum_{r=1}^{k}\gamma_j^{(r+Q+1)}(t_{lw}-T_r)_+^Q\right]\cdot t_{lw}^{p-1}, \tag{2}$$

for $Q+1 < p \leq k+Q+1$

$$\frac{\partial U_j}{\partial \gamma_j^{(p)}} = 2\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}\left[-x_j^{(lw)} + \sum_{q=1}^{Q+1}\gamma_j^{(q)}t_{lw}^{q-1} + \sum_{r=1}^{k}\gamma_j^{(r+Q+1)}(t_{lw}-T_r)_+^Q\right]\cdot \\ \cdot\left(t_{lw}-T_{p-(Q+1)}\right)_+^Q, \tag{3}$$

(when we put $\mathbf{r}+Q+1=p$, then $\mathbf{r}=p-(Q+1)$).

Case A: $1 \leq p \leq Q+1$ shall lead to (see formula 2, without the multiplier 2 on the right side of the equation):

$$\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}\left[-x_j^{(lw)} + \sum_{q=1}^{Q+1}\gamma_j^{(q)}t_{lw}^{q-1} + \sum_{r=1}^{k}\gamma_j^{(r+Q+1)}(t_{lw}-T_r)_+^Q\right]\cdot t_{lw}^{p-1}$$
$$= \sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p-1}\left[\sum_{q=1}^{Q+1}\gamma_j^{(q)}t_{lw}^{q-1} + \sum_{r=1}^{k}\gamma_j^{(r+Q+1)}(t_{lw}-T_r)_+^Q - x_j^{(lw)}\right].$$

In the second sum in the square brackets we will change the summation index from $r\colon 1 \to k$ to $r\colon 1+(Q+1) \to k+(Q+1)$ with proper adjustments to the summed expression:

$$\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p-1}\left[\sum_{q=1}^{Q+1}\gamma_j^{(q)}t_{lw}^{q-1} + \sum_{r=1+(Q+1)}^{k+(Q+1)}\gamma_j^{(r)}\left(t_{lw}-T_{r-(Q+1)}\right)_+^Q - x_j^{(lw)}\right] =$$

$$= \sum_{q=1}^{Q+1}\gamma_j^{(q)}\underbrace{\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p+q-2}}_{m_{pq}} + \sum_{r=Q+2}^{k+Q+1}\gamma_j^{(r)}\underbrace{\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p-1}\left(t_{lw}-T_{r-(Q+1)}\right)_+^Q}_{m_{pr}} - \underbrace{\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} x_j^{(lw)}t_{lw}^{p-1}}_{z_{pj}},$$

where:

$$m_{pq} = \begin{cases} \displaystyle\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p+q-2} & \text{for } A_1\colon 1 \leq q \leq Q+1, \\ \displaystyle\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} t_{lw}^{p-1}(t_{lw}-T_{q-(Q+1)})_+^Q & \text{for } A_2\colon Q+1 < q \leq k+Q+1, \end{cases} \tag{4}$$

and

$$z_{pj} = \sum_{l=1}^{k+1}\sum_{w=1}^{n(l)} x_j^{(lw)}t_{lw}^{p-1}. \tag{5}$$

Case B: $Q+1 < p \leq k+Q+1$ shall analogously lead to (see formula 3):

$$\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}\left[-x_j^{(lw)}+\sum_{q=1}^{Q+1}\gamma_j^{(q)}t_{lw}^{q-1}+\sum_{r=1}^{k}\gamma_j^{(r+Q+1)}(t_{lw}-T_r)_+^Q\right]\cdot\left(t_{lw}-T_{p-(Q+1)}\right)_+^Q=$$
$$=\sum_{q=1}^{Q+1}m_{pq}\gamma_j^{(q)}+\sum_{r=Q+2}^{k+Q+1}m_{pr}\gamma_j^{(r)}-z_{pj},$$

where:

$$m_{pq}=$$
$$=\begin{cases}\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}t_{lw}^{q-1}(t_{lw}-T_{p-(Q+1)})_+^Q & \text{for } B_1: 1\leq q\leq Q+1,\\ \sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}(t_{lw}-T_{q-(Q+1)})_+^Q(t_{lw}-T_{p-(Q+1)})_+^Q & \text{for } B_2: Q+1<q\leq k+Q+1,\end{cases} \tag{6}$$

and

$$z_{pj}=\sum_{l=1}^{k+1}\sum_{w=1}^{n(l)}x_j^{(lw)}\left(t_{lw}-T_{p-(Q+1)}\right)_+^Q. \tag{7}$$

Example 1.2

Let $k = 2$, $Q = 3$, hence $k + Q + 1 = 2 + 3 + 1 = 6$. We are to determine m_{pq} for $p = 3$, $q = 6$, thus m_{36}, and also $z_{pj} = z_{3j}$. It corresponds to Case A together with A_2 hence, according to (4):

$$m_{36}=\sum_{l=1}^{3}\sum_{w=1}^{n(l)}t_{lw}^2(t_{lw}-T_2)_+^3$$

and, according to (5):

$$z_{3j}=\sum_{l=1}^{3}\sum_{w=1}^{n(l)}x_j^{(lw)}t_{lw}^2.$$

Further, we are to determine m_{pq} for $p = 6$, $q = 3$, that is m_{63}, and also $z_{pj} = z_{6j}$. It corresponds to Case B together with B_1 hence, according to (6):

$$m_{63}=\sum_{l=1}^{3}\sum_{w=1}^{n(l)}t_{lw}^2(t_{lw}-T_2)_+^3$$

and, according to (7):

$$z_{6j}=\sum_{l=1}^{3}\sum_{w=1}^{n(l)}x_j^{(lw)}(t_{lw}-T_2)_+^3.$$

It holds that $m_{36} = m_{63}$.

Furthermore, we are to determine m_{pq} for $p = q = 5$, that is m_{55}, and also $z_{pj} = z_{5j}$. It corresponds to Case B together with B_2 hence, according to (6):

$$m_{55} = \sum_{l=1}^{3} \sum_{w=1}^{n(l)} (t_{lw} - T_1)_+^3 (t_{lw} - T_1)_+^3$$

and, according to (7):

$$z_{3j} = \sum_{l=1}^{3} \sum_{w=1}^{n(l)} x_j^{(lw)} t_{lw}^2.$$

Lastly, we are to determine m_{11} It corresponds to Case A together with A_1 hence, according to (4):

$$m_{11} = \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} t_{lw}^0 = \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} 1 = \sum_{l=1}^{k+1} n(l) = n.$$

This result holds for $k \geq 1$ arbitrary, independently of $Q \in \{1, 2, 3\}$.

We proceed in the minimization of U_j It is known from the general theory of mathematical analysis that a necessary condition for U_j as a function of the parameters $\gamma_j^{(1)}, \gamma_j^{(2)}, \ldots, \gamma_j^{(k+Q+1)}$, to attain its minimum is the system of equations $\frac{\partial U_j}{\partial \gamma_j^{(p)}} = 0$ for $p = 1, \ldots, k + Q + 1$ (see (2), (3)). Based on the aforementioned results (see Cases A and B), we arrive at the system of $k + Q + 1$ linear equations for the estimates $c_j^{(1)}, c_j^{(2)}, \ldots, c_j^{(k+Q+1)}$:

$$\boldsymbol{M}\boldsymbol{c}_j = \boldsymbol{Z}_j, \tag{8}$$

where: $\boldsymbol{M} = (m_{pq})_{1 \leq p,q \leq k+Q+1}$ is a $(k + Q + 1) \times (k + Q + 1)$ matrix, and $\boldsymbol{Z}_j = (z_{pj})_{1 \leq p \leq k+Q+1}$ and $\boldsymbol{c}_j = \left(c_j^{(p)}\right)_{1 \leq p \leq k+Q+1}$ are $(k + Q + 1) \times 1$ matrices. The equations of the system (8) are called normal equations. We can easily verify that $\boldsymbol{M}$ is a symmetric matrix.

By solving the system of equations (8) we get the sought estimates $c_j^{(1)}, c_j^{(2)}, \ldots, c_j^{(k+Q+1)}$ of the parameters $\gamma_j^{(1)}, \gamma_j^{(2)}, \ldots, \gamma_j^{(k+Q+1)}$ in the linear combination $g_j(t)$, see (1), $\in \langle T_0, T_{k+1} \rangle$. To these estimates we get the corresponding regression splines (linear for Q = 1, quadratic for Q = 2, and cubic for Q = 3) for $t \in \langle T_0, T_{k+1} \rangle$ through the equation:

$$x_j = G_j(t) = c_j^{(1)} + c_j^{(2)} t + \cdots + c_j^{(Q+1)} t^Q + \sum_{r=1}^{k} c_j^{(r+Q+1)} (t - T_r)_+^Q. \tag{9}$$

To sum up, (for $j = 1, \ldots, m$) these equations represent the parametric expression of a curve in $\mathbb{R}^m = (0; x_1, x_2, \ldots, x_m)$ that is the output of the regression model for the observed process. In short, we shall call it a regression curve (linear for Q = 1, quadratic for Q = 2, and cubic for = 3).[2]

[2] Literature: Spline functions – Bézier (1972), Böhmer (1974), Kaňka (2015), Makarov, Chlobystov (1983), Schmitter, Delgado-Gonzalo, Unser (2016), Sung (2016), Spät (1996), Vasilenko (1983), Wang, Wu, Shen, Zhang, Mou, Zhou (2016); Cut-off polynomials – Meloun, Militký (1994); Segmented regression – Feder (1975), Guzik (1974), Seger (1988).

2 POLYGONAL METHOD

For greater clarity, we shall confine ourselves to the plane $\mathbb{R}^2$.

In $\mathbb{R}^2$ let us consider the connected oriented graph $\vec{G} = [A,\vec{B}]$, with the set of vertices $A = \{1,2,3, ..., n\}$, $n \geq 2$, $\vec{B} = \{(1,2),(2,3), ..., (n-1,n)\}$ is the set of the (oriented) edges. We can imagine that the planar polygonal trail created like this and having its initial point in 1 and end point in n represents an idealized route of a car moving at a constant speed that started from place 1 and ended the journey at place n. Each vertex of the graph can be regarded as an experimental point, the coordinates of which are obtained by measuring its distance (e.g. in km) from the left and bottom edge of the map. We divide the graph vertices into $k+1$ groups ($k \geq 1$) by $n(l) \geq 1$ points $x_j^{(lw)}$ ($l = 1, ..., k+1$; $w = 1, ..., n(l)$; $j = 1,2$) in such a way that it holds that:

$$n = \sum_{l=1}^{k+1} n(l)$$

(such a division might be caused, e.g., by the difficulty of the corresponding road terrain), and assign to them (increasing) values t_{lw} (in km), where t_{lw} denotes the length of the passed route from the start 1 to the place $x_j^{(lw)}$, which could be regarded as a spot for a short break.

For $l = 1, ..., k+1$ we place the values $t_{l1} < t_{l2} < ... < t_{l,n(l)}$ into the interval $\langle T_{l-1}, T_l)$. Meanwhile, we will demand that the sequence of points $T_0, T_1, ..., T_k, T_{k+1}$ is increasing and it holds that $T_{l-1} \leq t_{l1}$ for $l = 1, ..., k+1$($T_1, ..., T_k$ are called main and the points $T_0 < T_1$ and $T_k + 1 > T_k$ complementary knots for the observed drive; compare with Section 1).

It gives sense to set $T_0 = 0$ then further it follows from the relation $T_l \leq t_{l+1,1}$ after substituting for $T_l = \lfloor t_{l,n(l)} \rfloor + p_l$ the inequality $\lfloor t_{l,n(l)} \rfloor + p_l \leq t_{l+1,1}$, thus $p_l \leq t_{l+1,1} - \lfloor t_{l,n(l)} \rfloor$. Let:

$$P = \min_{l=1,\dots,k} \{t_{l+1,1} - \lfloor t_{l,n(l)} \rfloor\} \tag{10}$$

and $p = \lfloor P \rfloor$ (by the symbol $\lfloor P \rfloor$ we shall understand the integer part of the corresponding real number). If $p \geq 1$, then we set $p_l = p$; we shall return to the case $p = 0$.

If we disregard the car drive example, we can say that there exists a certain automation for "operating" variable value assignment to experimentally obtained points divided in a certain way into groups, followed by the computation of so-called knots which define the interval of the assigned variable to the given group. We shall call this automation in short the "polygonal method". This procedure is implemented in the computer program TRIO, which is able to solve regression problems for $Q \in \{1, 2, 3\}$, in the plane $\mathbb{R}^2$ and in the space $\mathbb{R}^3$ as well.

Example 2.1

Let us consider in $\mathbb{R}^3$ the points $x_j^{(lw)}$ ($l = 1,2,3$; $w = 1, ..., n(l)$, where $n(1) = 2$, $n(2) = 3$, $n(3) = 2$; $j = 1,2,3$), divided into three groups:

$$x_j^{(11)} = (1, 1, 1), \qquad x_j^{(12)} = (1.1, 1.2, 1.3),$$

$$x_j^{(21)} = (1.5, 1, 1.4), \qquad x_j^{(22)} = (2, 3, 4), \qquad x_j^{(23)} = (3, 3, 5),$$

$$x_j^{(31)} = (3.1, 3.2, 5.05), \qquad x_j^{(32)} = (4, 4, 6).$$

According to the polygonal method, we assign to them (increasing) values of an operating variable (for example, time):

$$t_{11} = 0.0000, \qquad t_{12} = 0.3742,$$

$$t_{21} = 0.8325, \qquad t_{22} = 4.1506, \qquad t_{23} = 5.5648,$$

$$t_{31} = 5.7939, \qquad t_{32} = 7.3277.$$

According to (10), there is $P = \min\{0.8325, 0.7939\} = 0.7939$, hence $p = \lfloor P \rfloor = 0$.

We proceed further as follows. We substitute $x_j^{(lw)}$ with points $\tilde{x}_j^{(lw)} = L \cdot x_j^{(lw)}$, where L > 1 is sufficiently large, and assign to them through the polygonal method (increasing) values of an operating variable $\tilde{t}_{lw} = 10 \cdot t_{lw}$. In this way we obtain $P = \min\{5.3243, 2.9390\}$, thus $p = \lfloor P \rfloor = 2$. We get the knots $\tilde{T}_0 = 0, \tilde{T}_1 = 5, \tilde{T}_2 = 57, \tilde{T}_3 = 75$, which correspond to the initial knots $T_0 = 0, T_l = \frac{\tilde{T}_l}{L} (L = 10)$, that is, the points $T_0, T_1 = 0.5, T_2 = 5.7, T_3 = 7.5$, through which we carry out the sought segmented regression. The program TRIO has this procedure built in.

3 TRANSFORMATION OF THE PARAMETRIC VARIABLE

The elements of the matrix $\boldsymbol{M}$ and the vector Z_j of the system of normal equations (see (8)) are structured in such a way that reflects the fact that we are working with splines based on cut-off polynomials. To increase the numerical stability of the parametric equations of the regression curve (see (9)), which is the output of the given regression mode, the scientific literature proposes to implement a transformation of the corresponding parameter into a, for example, unit-length interval (if the length of the initial interval is greater than 1).

Let us deal with such a transformation in the case of, for example, quadratic regression, i.e., when $Q = 2$. Let us write the matrix of the system of normal equations $\boldsymbol{M} = (m_{pq})_{1 \le p,q \le k+3}$ in the following form:

$$\boldsymbol{M} = \begin{pmatrix} n & m_{12} & \boldsymbol{A} \\ m_{12} & m_{22} & \boldsymbol{B} \\ \boldsymbol{A}^T & \boldsymbol{B}^T & \boldsymbol{D} \end{pmatrix},$$

where:

$$n = m_{11},$$

$$\boldsymbol{A} = \left(m_{13}, m_{14}, \dots, m_{1,k+3}\right),$$

$$\boldsymbol{B} = \left(m_{23}, m_{,24}, \dots, m_{2,k+3}\right),$$

$\boldsymbol{A}^T$ is the transpose matrix of$\boldsymbol{A}$,

$\boldsymbol{B}^T$ is the transpose matrix of$\boldsymbol{B}$,

$$\boldsymbol{D} = \begin{pmatrix} m_{33} & m_{34} & \cdots & m_{3,k+3} \\ m_{34} & m_{44} & \cdots & m_{4,k+3} \\ \vdots & \vdots & \ddots & \vdots \\ m_{3,k+3} & m_{4,k+3} & \cdots & m_{k+3,k+3} \end{pmatrix},$$

and let us write the $k + 3$-dimensional vector $\boldsymbol{Z}_j = (z_{pj})_{1 \le p \le k+3}$ as:

$$\boldsymbol{Z}_j = \left(z_{1j}, z_{2j}, \left(z_{3j}, z_{4j}, \dots, z_{k+3,j}\right)\right) = \left(z_{1j}, z_{2j}, \boldsymbol{E}_j\right).$$

If we subject every value t_{lw} ($l = 1, ..., k + 1; w = 1, ..., n(l)$), including the knots $T_0, T_1, ..., T_k, T_{k+1}$, to the transformation

$$\begin{aligned} t'_{lw} &= K t_{lw}, \\ T'_0 &= K T_0, \qquad T'_l = K T_l, \end{aligned} \tag{11}$$

where $K = 1/(T_{k+1} - T_0)$, the matrix $\boldsymbol{M}$ transforms into:

$$\boldsymbol{M}' = \begin{pmatrix} n & Km_{12} & K^2\boldsymbol{A} \\ Km_{12} & K^2 m_{22} & K^3\boldsymbol{B} \\ K^2\boldsymbol{A}^T & K^3\boldsymbol{B}^T & K^4\boldsymbol{D} \end{pmatrix}$$

and $\boldsymbol{Z}_j$ transforms into $\boldsymbol{Z}'_j = (z_{1j}, Kz_{2j}, K^2\boldsymbol{E}_j)$. Now, if the (k + 3) – dimensional vector:

$$\boldsymbol{c}_j = \left(c_j^{(p)}\right) = \left(c_j^{(1)}, c_j^{(2)}, c_j^{(3)}, c_j^{(4)}, \dots, c_j^{(k+3)}\right) = \left(c_j^{(1)}, c_j^{(2)}, \boldsymbol{C}_j\right)$$

is the solution of the system of normal equations, the following equality holds:

$$\begin{pmatrix} n & m_{12} & \boldsymbol{A} \\ m_{12} & m_{22} & \boldsymbol{B} \\ \boldsymbol{A}^T & \boldsymbol{B}^T & \boldsymbol{D} \end{pmatrix} \begin{pmatrix} c_j^{(1)} \\ c_j^{(2)} \\ \boldsymbol{C}_j^T \end{pmatrix} = \begin{pmatrix} z_{1j} \\ z_{2j} \\ \boldsymbol{E}_j^T \end{pmatrix},$$

which is equivalent to the equality:

$$\begin{pmatrix} n & Km_{12} & K^2\boldsymbol{A} \\ Km_{12} & K^2 m_{22} & K^3\boldsymbol{B} \\ K^2\boldsymbol{A}^T & K^3\boldsymbol{B}^T & K^4\boldsymbol{D} \end{pmatrix} \begin{pmatrix} c_j^{(1)} \\ c_j^{(2)}/K \\ \boldsymbol{C}_j^T/K^2 \end{pmatrix} = \begin{pmatrix} z_{1j} \\ Kz_{2j} \\ K^2\boldsymbol{E}_j^T \end{pmatrix}.$$

Hence, it follows that the vector $t' \in \langle KT_0, KT_{k+1} \rangle$, meets the equality $\boldsymbol{M}'_j \boldsymbol{c}'_j = \boldsymbol{Z}'_j$. The equation for $t' \in \langle KT_0, KT_{k+1} \rangle$, of the quadratic regression spline with this vector is the cut-off polynomial:

$$x_j = G'_j(t') = c_j^{(1)} + \frac{1}{K} c_j^{(2)} t' + \frac{1}{K^2} c_j^{(3)} (t')^2 + \frac{1}{K^2} \sum_{r=1}^{k} c_j^{(r+3)} (t' - KT_r)_+^2. \tag{12}$$

The equation (12) for $t' \in \langle KT_0, KT_{k+1} \rangle$, represents the same regression curve in $\mathbb{R}^m = (0; x_1, x_2, ..., x_m)$ as the equation with the untransformed parameter:

$$x_j = G_j(t) = c_j^{(1)} + c_j^{(2)} t + c_j^{(3)} t^2 + \sum_{r=1}^{k} c_j^{(r+3)} (t - T_r)_+^2, \qquad t' \in \langle T_0, T_{k+1} \rangle.$$

We obtain analogous results for the case when $Q = 1$ or $Q = 3$, as well.

4 SOLUTION TO PARTICULAR PROBLEMS

Example 4.1

We are supposed to solve a problem in which we reflect the aforementioned procedures. Our results were obtained by the computer program TRIO, created by the author of this article for the purposes of segmented regression, without which particular computations would be unfeasible by hand.

Table 1 Meteorological data

Time [h]		Temperature	Pressure	Wind
Real	Fictive	[°C]	[hPa]	[m/s]
6	1	15	800	5
8	2	16	850	4
10	3	17	900	3
12	4	22	1 000	2
14	5	28	1 030	1
16	6	26	1 020	2
18	7	20	950	3
20	8	19	900	3
22	9	18	890	3
24	10	16	840	4
2	11	15	820	4
4	12	13	810	5

Source: M. Kaňka

We assume that over time period from 6 am the values of the following three indicators: air temperature, air pressure and wind speed. The result of this measurement can be seen in Table 1.

In $\mathbb{R}^3$ (hence, $j = 1, 2, 3$) we have 12 experimentally obtained points, split in four groups (hence, $k = 3$) by the three points (we may call them morning, noon, evening and night group):

$$\begin{array}{lll} x_j^{(11)} = (15, 800, 5), & x_j^{(12)} = (16, 850, 4), & x_j^{(13)} = (17, 900, 3), \\ x_j^{(21)} = (22, 1000, 2), & x_j^{(22)} = (28, 1030, 1), & x_j^{(23)} = (26, 1020, 2), \\ x_j^{(31)} = (20, 950, 3), & x_j^{(32)} = (19, 900, 3), & x_j^{(33)} = (18, 890, 3), \\ x_j^{(41)} = (16, 840, 4), & x_j^{(42)} = (15, 820, 4), & x_j^{(43)} = (13, 810, 5), \end{array} \tag{13}$$

to which we assign the following values of a fictitious time:

$$\begin{array}{llll} t_{11} = 1, & t_{12} = 2, & t_{13} = 3, & \text{thus } n(1) = 3, \\ t_{21} = 4, & t_{22} = 5, & t_{23} = 6, & \text{thus } n(2) = 3, \\ t_{31} = 7, & t_{32} = 8, & t_{33} = 9, & \text{thus } n(3) = 3, \\ t_{41} = 10, & t_{42} = 11, & t_{42} = 12, & \text{thus } n(4) = 3, \end{array} \tag{14}$$

it holds that $\sum_{l=1}^{k+1} n(l) = \sum_{l=1}^{4} n(l) = 3 + 3 + 3 + 3 = 12$, which is the total number of considered points the values of which may be $T_1 = 4$, $T_2 = 7$, $T_3 = 10$, together, for example, with additional time moments $T_0 = 1$ and $T_4 = 13$.

For example, for $Q = 1$, the matix $\boldsymbol{M}$ of system (8) is a $(k + Q + 1) \times (k + Q + 1) = 5 \times 5$ matrix. To save space, we neither give its full expression, nor for the 5-dimensional vectors $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_3$ on the right-hand side of this equation. This computationally intensive work was conducted by the computer program TRIO, that the author of this article created for the purposes of segmented regression on the basis of B-splines.

Nevertheless, for demonstration purposes, let us compute the element m_{43} of $\boldsymbol{M}$ and the element z_{53} of $\boldsymbol{Z}_3$. As $Q + 1 = 2 < 3 = q \le k + Q + 1 = 5$, there will be, according to (6):

$$m_{43} = \sum_{l=1}^{4}\sum_{w=1}^{3}(t_{lw} - T_1)_+ \cdot (t_{lw} - T_2)_+ =$$

$$= \sum_{w=1}^{3}(t_{1w} - 4)_+ \cdot (t_{1w} - 7)_+ + \sum_{w=1}^{3}(t_{2w} - 4)_+ \cdot (t_{2w} - 7)_+ +$$

$$+ \sum_{w=1}^{3}(t_{3w} - 4)_+ \cdot (t_{3w} - 7)_+ + \sum_{w=1}^{3}(t_{4w} - 4)_+ \cdot (t_{4w} - 7)_+ =$$

$$= 0 + 0 + (7-4)_+ \cdot (7-7)_+ + (8-4)_+ \cdot (8-7)_+ + (9-4)_+ \cdot (9-7)_+ +$$

$$+(10-4)_+ \cdot (10-7)_+ + (11-4)_+ \cdot (11-7)_+ + (12-4)_+ \cdot (12-7)_+ =$$

$$= 0 + 0 + 0 + 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 + 7 \cdot 4 + 8 \cdot 5 = 100 = m_{34},$$

as $\boldsymbol{M}$ is symmetric, and further:

$$z_{53} = \sum_{l=1}^{4}\sum_{w=1}^{3} x_3^{(lw)}(t_{lw} - T_3)_+ =$$

$$= \sum_{w=1}^{3} x_3^{(1w)}(t_{1w} - 10)_+ + \sum_{w=1}^{3} x_3^{(2w)}(t_{2w} - 10)_+ +$$

$$+ \sum_{w=1}^{3} x_3^{(3w)}(t_{3w} - 10)_+ + \sum_{w=1}^{3} x_3^{(4w)}(t_{4w} - 10)_+ =$$

$$= 0 + 0 + 0 + x_3^{(41)}(t_{41} - 10)_+ + x_3^{(42)}(t_{42} - 10)_+ + x_3^{(43)}(t_{43} - 10)_+ =$$

$$= 0 + 0 + 0 + 4 \cdot (10-10)_+ + 4 \cdot (11-10)_+ + 5 \cdot (12-10)_+ =$$

$$= 0 + 0 + 0 + 0 + 4 \cdot 1 + 5 \cdot 2 = 14.$$

The computer program TRIO provides the parametric equations of the resulting (for our case $Q = 1$, linear) regression curve that we do not present here to save space.

The computation of the so-called determination indices, which is also provided by the program TRIO, yields for the aforementioned experiment (where Q = 1) the values $I_{x_1}^2 = 0.7539, I_{x_2}^2 = 0.9296, I_{x_3}^2 = 0.9009$.

This means that approximately 75% variability ofthe observed values x_1, 93% variability of x_2 and 90% variability of x_3 can be explained bya linear regression model. For $Q = 2$ and $Q = 3$ the program TRIO gives the following determination indices, see Table 2.

Table 2 Determination indices describing variability of observed values

Regression	$I_{x_1}^2$	$I_{x_2}^2$	$I_{x_3}^2$
Q=1	0.7539	0.9296	0.9009
Q=2	0.9302	0.9748	0.9179
Q=3	0.8998	0.9774	0.9474

Source: M. Kaňka

With respect to the determination indices, one can consider a certain “optimal regression curve” the equation of which for x_1 is based on quadratic regression ($Q = 2$), the equation for x_2 and x_3 on cubic regression ($Q = 3$):

$$x_1 = G_1(t) = \begin{cases} 16.7449 - 2.6212t + 0.9998t^2 & \text{for } 1 \leq t < 4, \\ -27.0951 + 19.2988t - 1.7402t^2 & \text{for } 4 \leq t < 7, \\ 104.7835 - 18.3808t + 0.9512t^2 & \text{for } 7 \leq t < 10, \\ -103.7065 + 23.3172t - 1.1337t^2 & \text{for } 10 \leq t \leq 13, \end{cases}$$

$$x_2 = G_2(t) = \begin{cases} 897.9027 - 176.7071t + 93.6242t^2 - 10.7554t^3 & \text{for } 1 \leq t < 4, \\ -51.2941 + 535.1905t - 84.3502t^2 + 4.0758t^3 & \text{for } 4 \leq t < 7, \\ 1200.3472 - 1.2272t - 7.7191t^2 + 0.4267t^3 & \text{for } 7 \leq t < 10, \\ 1380.2472 - 55.1972t - 2.3221t^2 + 0.2468t^3 & \text{for } 10 \leq t \leq 13, \end{cases}$$

$$x_3 = G_3(t) = \begin{cases} 4.0879 + 1.9577t - 1.2539t^2 + 0.1537t^3 & \text{for } 1 \leq t < 4, \\ 19.3391 - 9.4807t + 1.6057t^2 - 0.0846t^3 & \text{for } 4 \leq t < 7, \\ -20.2431 + 7.4831t - 0.8177t^2 + 0.0308t^3 & \text{for } 7 \leq t < 10, \\ -9.7491 + 4.3331t - 0.5027t^2 + 0.0203t^3 & \text{for } 10 \leq t \leq 13. \end{cases}$$

For example, for $t = 8$ we get the point (18.6139, 914.9777, 3.0585) on the optimal regression curve, which lies “near” the point (19, 900, 3) to which the value of the parameter belongs. Or for $t = 4.5$ we obtain the point (24.5105, 1020.3789, 1.4822) on the optimal regression curve. We may draw the conclusion that at 1 pm local time, the air temperature was approximately 25°C, the air pressure approximately 1020 hPa and the speed of the wind approximately 1 m/s.

Example 4.2

We shall solve the problem from Example 4.1 with the help of the polygonal method together with a transformation of the parametric variable (see Sections 2, 3). We assign to the points (13), which were arranged into four groups ($k = 3$), values $\tilde{t}$:

$$\tilde{t}_{11} = 0.00, \quad \tilde{t}_{12} = 50.02, \quad \tilde{t}_{13} = 100.04, \quad \text{thus } n(1) = 3,$$

$$\tilde{t}_{21} = 200.17, \quad \tilde{t}_{22} = 230.78, \quad \tilde{t}_{23} = 241.03, \quad \text{thus } n(2) = 3,$$

$$\tilde{t}_{31} = 311.29, \quad \tilde{t}_{32} = 361.30, \quad \tilde{t}_{33} = 371.35, \quad \text{thus } n(3) = 3,$$

$$\tilde{t}_{41} = 421.40, \quad \tilde{t}_{42} = 441.42, \quad \tilde{t}_{43} = 451.67, \quad \text{thus } n(4) = 3.$$

We easily find out that the number $\tilde{t}_{lw}$ expresses the length of the polygonal trail measured from the initial point $x_j^{(11)}$ to the considered point $x_j^{(lw)}$ ($l = 1,2,3,4$; w = 1,2,3), see (13). According to (10), in this case there is:

$$P = \min_{l=1,2,3}\{\tilde{t}_{l+1,1} - \lfloor\tilde{t}_{l,n(l)}\rfloor\} = \min\{100.17, 70.29, 50.4\},$$

thus p = ⌊P⌋ = 50, hence the knots of the problem are $\tilde{T}_0 = 0, \tilde{T}_1 = \lfloor 100.04 \rfloor + 50 = 150$, $\tilde{T}_2 = \lfloor 241.03 \rfloor + 50 = 291$, $\tilde{T}_3 = \lfloor 371.35 \rfloor + 50 = 421$, $\tilde{T}_4 = \lfloor 451.67 \rfloor + 50 = 501$. Through the transformation $t' = K\tilde{t}$, where $K = 1/(\tilde{T}_{k+1} - \tilde{T}_0) = 1/(\tilde{T}_4 - \tilde{T}_0) = 1/501$, the latter values of $\tilde{t}_{lw}$ are mapped onto t'_{lw}, $= K\tilde{t}_{lw} = \tilde{t}_{lw}/501$ and the knots $\tilde{T}_0 = 0, \tilde{T}_1 = 150$, $\tilde{T}_2 = 291$, $\tilde{T}_3 = 421$, $\tilde{T}_4 = 501$ onto $T'_0 = 0, T'_1 = 0.3$, $T'_2 = 0.58, T'_3 = 0.84$, $T_4 = 1$. Then, a subsequent execution of regression for Q = 1,2,3 provides (through the computer program TRIO) the following table of determination indices:

Table 3 Determination indices describing variability of observed values

Regression	$I^2_{x_1}$	$I^2_{x_2}$	$I^2_{x_3}$
Q=1	0.8017	0.9584	0.9009
Q=2	0.9193	0.9953	0.9308
Q=3	0.9004	0.99	0.9307

Source: M. Kaňka

With respect to the measures of the determination indices, it can be seen that the equations of the optimal regression curve for x_1,x_2,x_3 are according to the quadratic regression (Q = 2):

$$x_1 = G_1'(t') = \begin{cases} 15.3435 - 12.2933t' + 107.2952(t')^2 & \text{for } 0 \le t' < 0.3, \\ -9.0168 + 150.4338t' - 164.4590(t')^2 & \text{for } 0.3 \le t' < 0.58, \\ 66.5740 - 109.8479t' + 59.5979(t')^2 & \text{for } 0.58 \le t' < 0.84, \\ -476.4334 + 1182.5355t' - 709.3856(t')^2 & \text{for } 0.84 \le t' \le 1, \end{cases}$$
$$x_2 = G_2'(t') = \begin{cases} 801.6375 + 393.7033t' + 516.9006(t')^2 & \text{for } 0 \le t' < 0.3, \\ 546.2165 + 2099.9158t' - 2332.4744(t')^2 & \text{for } 0.3 \le t' < 0.58, \\ 1431.8960 - 949.7437t' + 292.7480(t')^2 & \text{for } 0.58 \le t' < 0.84, \\ 700.7491 + 790.4207t' - 742.6705(t')^2 & \text{for } 0.84 \le t' \le 1, \end{cases} \tag{15}$$
$$x_3 = G'_3(t') = \begin{cases} 5.0080 - 10.8967t' + 5.2201(t')^2 & \text{for } 0 \le t' < 0.3, \\ 7.0535 - 24.5605t' + 28.0387(t')^2 & \text{for } 0.3 \le t' < 0.58, \\ -5.6486 + 19.1765t' - 9.6112(t')^2 & \text{for } 0.58 \le t' < 0.84, \\ 193.8010 - 455.5231t' + 272.8407(t')^2 & \text{for } 0.84 \le t' \le 1. \end{cases}$$

E.g., for $t'_{32} = \tilde{t}_{32}/501 = 361.30/501 = 0.7212$ we get (18.3503, 899.2077, 3.1824) on the optimal regression curve that lies "near" the point $(19,900,3) = t_{32}$, see (13). Or for $t'_{43} = \tilde{t}_{43}/501 = 451.67/501 = 0.9015$ we obtain the point (13.1031, 809.7434, 4.8852) that lies "near" the point $(13,810,5) = t_{43}$, see (13).

We might also be interested in the question how to determine for $t = 4.5$, which lies between $t_{21} = 4$ and $t_{22} = 5$, see (14), the value t' lying between $t'_{21}/501 = 200.17/501 = 0.3995$ and $t'_{22}/501 = 230.78/501 = 0.4606$. Through the function:

$$\tilde{t} = \tilde{t}_{22} + \frac{\tilde{t}_{22} - \tilde{t}_{21}}{5 - 4}(t - 5),$$

that maps the interval $\langle 4,5 \rangle$ for the variable t onto $\langle \tilde{t}_{21}, \tilde{t}_{22} \rangle$ for $\tilde{t}$ we obtain for $t = 4.5$ the value:

$$\tilde{t} = 230.78 + \frac{230.78 - 200.17}{1}(-0.5) = 215.4750.$$

The transformation of the interval $\langle \tilde{T}_0 = 0, \tilde{T}_4 = 1 \rangle$ onto $\langle 0,1 \rangle$ is then done through the function:

$$t' = 1 + \frac{1 - 0}{501 - 0}(\tilde{t} - 501),$$

which yields then after the substitution $\tilde{t} = 215.4750$ that:

$$t' = 1 + \frac{1}{501}(215.4750 - 501) = 0.4306.$$

Inserting this value into (15), we obtain on the optimal regression curve that is, in comparison with the data in Table 1, quite acceptable.

CONCLUSION

Author was paying close attention to so called polygonal method of assigning values of parametric variable to experimentally obtained points, which displays those points with an oriented graph, and as a parameter of considered point it selects the length of polygonal trail measured from the initial point to the considered point. The main result of this procedure involves automatic computation of nodal points, which are associated with given task.

In order to improve numerical stability of equations of regressive curve, which is associated with the solution of specific task, it is recommended in literature to transform parametric interval to an interval of length one.

At the end of the paper the author introduces term optimal regression, which takes into account values of indexes of determination of observed units in specific model ($Q = 1,2,3$).

Individual stages of solution concerning the task of segmented regression are demonstrated in an example from meteorology. Introduced computations are product of computer program TRIO, without which the computations (starting with the elements of matrix of system of normal equations) would be hardly feasible.

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Conferences

The European Conference on Quality in Official Statistics (Q2016) took place in **Madrid, Spain, from 1st to 3rd June 2016.** The conference **was** organized by the National Statistical Institute of Spain (INE) and Eurostat and aims to cover relevant and innovative topics on quality ranging from the challenges and the new paradigm of quality in an information and knowledge-driven society including big data and multi-source statistics, to governance and management aspects like the ones linked to the ESS Vision 2020 or the lessons learned from 2013–2015 peer reviews in the European Statistical System. More information available at: *http://www.q2016.es*.

The 22nd International Conference on Computational Statistics (COMPSTAT 2016) will take place at the Conference Centre of **Oviedo, Spain, during 23–26 August 2016**. The conference aims at bringing together researchers and practitioners to discuss recent developments in computational methods, methodology for data analysis and applications in statistics. The conference is organized by the University of Oviedo. More information available at: *http://www.compstat2016.org*.

The 19th International Scientific Conference Applications of Mathematics and Statistics in Economics (AMSE 2016) will be held in **Banska Stiavnica, Slovakia, from 31st August to 4th September 2016.** These conferences are organized each year by three Faculties of three Universities from three countries – University of Economics, Prague, Czech Republic, Matej Bel University in Banska Bystrica, Slovakia, and Wrocław University of Economics, Poland. More information available at: *http://amse.umb.sk*.

The 34th International Conference on Mathematical Methods in Economics (MME 2016) will take place in **Liberec, Czech Republic, during 6–9 September 2016**. The conference is a traditional meeting of professionals from universities and businesses interested in the theory and applications of operations research and econometrics. Conference is held under auspices of the Technical University of Liberec, Czech Republic, Czech Society for Operations Research, Slovak Society for Operations Research and the Czech Econometric Society. More information available at: *http://mme2016.tul.cz*.

The 24th Interdisciplinary Information Management Talks (IDIMT 2016) will be held in **Poděbrady, Czech Republic, from 7th to 9th September 2016**. More information available at: *http://www.idimt.org*.

The 10th International Days of Statistics and Economics (MSED 2016) will take place in **Prague, Czech Republic, during 8–10 September 2016**, organized by the Department of Statistics and Probability and the Department of Microeconomics, University of Economics, Prague, Czech Republic, Faculty of Economics, Technical University of Košice, Slovakia, and Ton Duc Thang University, Vietnam. The aim of the conference is to present and discuss current problems of statistics, demography, economics and management and their mutual interconnection. More information available at: *https://msed.vse.cz*.

Papers
We publish articles focused at theoretical and applied statistics, mathematical and statistical methods, conception of official (state) statistics, statistical education, applied economics and econometrics, economic, social and environmental analyses, economic indicators, social and environmental issues in terms of statistics or economics, and regional development issues.

The journal of *Statistika* has the following sections:
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