A New Viterbi-Based Decoding Strategy for Market Risk Tracking: an Application to the Tunisian Foreign Debt Portfolio During 2010–2012

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Abstract

In this paper, a novel market risk tracking and prediction strategy is introduced. Our approach takes volatility clustering into account and allows for the possibility of regime shifts in the intra-portfolio's latent correlation structure. The proposed specification combines hidden Markov models (HMM) with latent factor models that takes into account the presence of both the conditional skewness and leverage effects in stock returns.

A computationally efficient expectation-maximization (EM) algorithm based on the Viterbi decoder is developed to estimate the model parameters. Using daily exchange rate data of the Tunisian dinar versus the currencies of the main Tunisian government's creditors, during the 2011 revolution period, the model parameters are estimated. Then, the suitable model is used in conjunction with a Monte Carlo simulation strategy to predict the Value-at-Risk (VaR) of the Tunisian government's foreign debt portfolio. The backtesting results indicate that the new approach appears to give a good fit to the data and can improve the VaR predictions, particularly during financial instability periods.

Keywords	***DOI***	***JEL code***
Factor analysis, volatility clustering, hidden Markov models, Viterbi-EM algorithm, portfolio's Value-at-Risk	*https://doi.org/10.54694/stat.2022.17*	*C38, C53, G17, G32*

INTRODUCTION

According to Saidane (2017) and Mosbahi et al. (2017), the understanding of co-movements among asset returns is a central element in the portfolio risk management process. The authors advocate the use of a mixture of probabilistic factor analyzers and the conditionally heteroskedastic latent factor model to handle co-movements, heterogeneity and time-varying volatility embedded in financial data. They

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demonstrate how their proposed strategies can be applied to the estimation of the portfolio's Value-at-Risk (VaR). However, an assumption of these models is that the correlation structure of the portfolio is assumed to be constant over time, but recent empirical works (e.g. Saidane, 2019; Tsang and Chen, 2018; Hamilton, 2016; Ang and Timmermann, 2012) have shown that this assumption of structural stability is invalid for financial returns, especially during crisis periods. For example, when the economy is hit by a permanent or temporary exogenous unpredictable shock, the cross-correlation behavior among several financial assets and the inter-relationship between volatilities can be expected to shift simultaneously. In light of this, we propose a novel market risk prediction strategy considering the possibility of regime switching in the interrelationships among several asset classes.

The new approach presented in this paper allows for the possibility of regime shifts in the intra-portfolio's latent correlation structure and takes volatility clustering into account. The proposed specification combines latent factor models that takes into account the presence of both the conditional skewness and leverage effects with hidden Markov models (HMM). To capture the volatility clustering and the leverage effect patterns of the return series, we assume that the common variances are modeled separately using quadratic generalized autoregressive conditionally heteroskedastic (GQARCH) processes. This provides a more tractable way to handle the time-varying volatility, co-movements and the latent heterogeneity in financial data.

For the maximum likelihood estimation we proceed in two steps. In the first step, we use the Viterbi decoding algorithm to find the most probable path through the HMM, given the observed data, which we take as an estimate of the true path. In the second step, we implement the Expectation-Maximization (EM) algorithm introduced by Dempster et al. (1977), to estimate the model parameters. Our proposed estimation strategy overcomes the complexity and limitations of the exact learning algorithm, especially when the number of hidden states and the length of the time sequence become larger.

The remainder of this paper is organized as follows. In Section 1, we provide further background on the factorial hidden Markov volatility model. In section 2, we discuss the inference procedure for the latent factors structure. We then present our iterative maximum-likelihood expectation-maximization (EM) algorithm in Section 3. We describe the portfolio's VaR simulation-based Viterbi tracking strategy in Section 4 and report on the backtesting results in Section 5. In this paper, the currency risk of the Tunisian government's foreign debt portfolio during the revolution period of 14 January 2011 is considered as the basis for an application to our novel prediction strategy. Our portfolio includes the main debt currencies against the Tunisian dinar, such as the European euro, the American dollar, the Japanese yen, the Swiss franc and the British pound. Finally, we conclude the paper by summarizing our contributions and discussing the future research directions.

1 THE FACTORIAL HIDDEN MARKOV VOLATILITY MODEL

Throughout this paper, we consider a multivariate discrete-time model. The closing price of the k-th asset in the portfolio at the t-th trading day is denoted by $p_{k,t}$, and the opening price at the first trading day by $p_{k,0}$.

For each $t \geq 1$, let $r_{k,t} = \log(p_{k,t} / p_{k,t-1})$ be the log-return of the k-th asset. Our model assumes a Markov switching relationship between the observed variables (the log-returns) and a set of q latent factors, which depend on the market regime. This new framework, called factorial hidden Markov volatility model (FHMV), is defined by:

$$\boldsymbol{r}_t = \boldsymbol{\Phi}_j \boldsymbol{z}_t + \epsilon_t , \tag{1}$$

where: $\forall\ t = 1, \dots , T$, $\boldsymbol{r}_t$ is a $(p \times 1)$ vector of log-returns.

The transition probabilities of the first order homogenous hidden Markov process from state i to state j ($\forall\ i, j = 1, \dots , n$) are represented by $p(S_t = j \mid S_{t-1} = i)$, where j is the actual market regime at time t, given the previous regime i at time $t - 1$. In a specified regime $S_t = j$, $\boldsymbol{\Phi}_j$ is the $(p \times q)$ factor loadings matrix.

The common latent factors z_t are generated from the multivariate normal distributions:

$$z_t \sim N(\mathbf{0}, \mathbf{\Omega}_j)\ , \tag{2}$$

where: $\mathbf{0}$ and $\mathbf{\Omega}_j$ denote, respectively, the $(q \times 1)$ mean vectors and $(q \times q)$ diagonal covariance matrices of the latent vectors z_t.

The diagonal elements of $\mathbf{\Omega}_j$ (common variances) are described by switching univariate quadratic GARCH(1,1) processes. Under a particular regime $S_t = j$ since $S_{t-1} = i$, the l-th common factor variance is given by:

$$\omega^j_{l,t} = \beta^j_{0l} + \beta^j_{1l} z^i_{l,t-1} + \beta^j_{2l} z^{i2}_{l,t-1} + \beta^j_{3l} \omega^i_{l,t-1}\ . \tag{3}$$

Assuming that $\beta^j_{1l}, \beta^j_{2l} > 0$, if $z_{l,t-1} < 0$, its impact on the variance $\omega_{l,t}$ is lower than in the case where $z_{l,t-1} > 0$.

Finally, the $(p \times 1)$ vector of specific factors can be written as follows:

$$\epsilon_t \sim N(\boldsymbol{\mu}_j, \mathbf{\Lambda}_j)\ , \tag{4}$$

where: $\boldsymbol{\mu}_j$ and $\mathbf{\Lambda}_j$ are, respectively, the $(p \times 1)$ mean vectors and $(p \times p)$ diagonal covariance matrices of the specific factors.

Assumption 1: In order to insure the positivity of the common variances and the stationarity of the covariance structure of the studied series, we introduce some constraints on the parameters of the quadratic GARCH specification, such as: $\beta^j_{2l} + \beta^j_{3l} < 1$, $\beta^j_{0l}, \beta^j_{2l}, \beta^j_{3l} > 0$ and $\beta^{j2}_{1l} \leq 4\beta^j_{0l}\beta^j_{2l}$, $\forall_j = 1, \dots, n$, $l = 1, \dots, q$.

Assumption 2: To guarantee the model identification in (1), we assume that $\forall_j$, rank $(\mathbf{\Phi}_j) = q$ and $p \geq q$. The factors z_t and ϵ_t are also assumed to be uncorrelated and mutually independent. For more detailed discussions of the identification problem, the reader can refer to Saidane and Lavergne, (2011), and Carnero (2004).

2 INFERENCE OF THE LATENT FACTORS STRUCTURES

Our model can be expressed as a switching state-space system with a measurement equation:

$$\boldsymbol{r}_t = \boldsymbol{\mu}_j + \mathbf{\Phi}_j\, z_t + \epsilon_t\ , \tag{5}$$

and a transition equation:

$$z_t = \mathbf{0} \cdot z_{t-1} + z_t\ , \tag{6}$$

In order to find the optimal sequences of hidden states S_t and latent factors z_t, we can use the Viterbi decoding algorithm based on the minimization of the Hamiltonian cost function given by the following equation:

$$\begin{aligned} H(r_{1:T}, Z_{1:T}, S_{1:T}) \simeq\ & c + \boldsymbol{S}_1^{'}(-\log\boldsymbol{\pi}) + \sum_{t=2}^{T} \boldsymbol{S}_1^{'}(-\log\boldsymbol{P})\ \boldsymbol{S}_{t-1} + \frac{1}{2}\sum_{t=1}^{T}\sum_{j=1}^{n} [\log|\mathbf{\Lambda}_j| + (\boldsymbol{r}_t - \mathbf{\Phi}_j z_t - \boldsymbol{\mu}_j)^{'}\mathbf{\Lambda}_j^{-1} \\ & (\boldsymbol{r}_t - \mathbf{\Phi}_j\, z_t - \boldsymbol{\mu}_j)]\ S_t\,(\mathrm{j}) + \frac{1}{2}\sum_{t=1}^{T}\sum_{j=1}^{n} [\log|\mathbf{\Omega}_j| + z_t^{'}\mathbf{\Omega}_j^{-1} z_t]\ S_t\,(\mathrm{j}), \end{aligned} \tag{7}$$

where: $r_{1:\tau} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_\tau\}$, $Z_{1:\tau} = \{z_1, z_2, \dots, z_\tau\}$, $S_{1:\tau} = \{\boldsymbol{S}_1, \boldsymbol{S}_2, \dots, \boldsymbol{S}_\tau\}$ are, respectively, the sequences of observed returns, latent common factors and HMM states up to time τ; $\boldsymbol{\pi}$ the vector of initial state probabilities, of length n-states and summing to 1; $\boldsymbol{P}$ the matrix of transition probabilities of the hidden Markov chain, the sum of all elements in the i-th row $[p_{i1} \dots p_{in}]$ is 1, $\forall\ i = 1, \dots, n$ and $\boldsymbol{S}_t = [S_t(1), \dots, St(n)]'$, where $S_t\,(j) = 1$, if $S_t = j$ and 0 otherwise.

If we denote by $S^*_{1:T}$ the optimal sequence of HMM states, the posterior distribution $p(Z_{1:T}, S_{1:T} \mid r_{1:T})$ can be approximated as:

$$p(Z_{1:T}, S_{1:T} \mid r_{1:T}) \simeq \eta(S_{1:T} - S^*_{1:T})\, p(Z_{1:T} \mid S_{1:T}, r_{1:T})\,, \tag{8}$$

i.e. the posterior distribution of the HMM state sequence $p(S_{1:T} \mid r_{1:T})$ is approached by its mode, where $\eta(y) = 1$ for $y = \phi$ and zero otherwise. The optimal sequence of HMM states can formally be obtained by solving the dynamic optimization program: $S^*_{1:T} = \arg\max_{S_{1:T}} p(S_{1:T} \mid r_{1:T})$. In this case, an almost optimal solution can be reached by maximizing recursively the probability of the best HMM sequence up to time t:

$$\begin{aligned}\delta_{t,j} &= \max_{S_{1:t-1}} p(S_{1:t-1}, S_t = j, r_{1:t}) \simeq \max_i \{p(\boldsymbol{r}_t \mid S_t = j, S_{t-1} = i, S^*_{1:t-2}(i), r_{1:t-1})\, p(S_t = j \mid S_{t-1} = i) \\ &\times \max_{S_{1:t-2}} p(S_{1:t-2}), \mathrm{S}_{\,t-1} = \mathrm{i}, r_{1:t-1})\},\end{aligned} \tag{9}$$

where: $S^*_{1:t-2}(i) = \arg\max_{S_{1:t-2}} \delta_{t-1,i}$ is the "optimal" HMM sequence up to time $t-1$ when the market state is in regime i at time $t-1$.

Firstly we define the "optimal" partial Hamiltonian cost up to time t of the observed log-return sequence $r_{1:t}$ when the market state is in regime j at time t:

$$\delta_{t,j} = \min_{S_{1:t-1}, Z_{1:t}} H(Z_{1:t}, \{S_{1:t-1}, S_t = j\}, r_{1:t})\,. \tag{10}$$

To calculate this cost correctly, we need the optimal filtered estimates of the common latent factors $\boldsymbol{z}^j_{t|t} = \mathbb{E}[\boldsymbol{z}_t | r_{1:t}, S_t = j]$, the one-step ahead predictions of the common latent factors $\boldsymbol{z}^{i(j)}_{t|t-1} = \mathbb{E}[\boldsymbol{z}_t | r_{1:t-1}, S_t = j, S_{t-1} = i]$ and their optimal filtered estimates, $\boldsymbol{z}^{i(j)}_{t|t} = \mathbb{E}[\boldsymbol{z}_t | r_{1:t}, S_t = j, S_{t-1} = i]$. We need also the predicted and filtered common variances:

$$\boldsymbol{\Omega}^{i(j)}_{t|t-1} = \mathbb{E}[(\boldsymbol{z}_t - \boldsymbol{z}^{i(j)}_{t|t-1})\,(\boldsymbol{z}_t - \boldsymbol{z}^{i(j)}_{t|t-1})' \mid r_{1:t-1}, S_t = j, S_{t-1} = i], \tag{11}$$

$$\boldsymbol{\Omega}^{ij}_{t|t} = \mathbb{E}[(\boldsymbol{z}_t - \boldsymbol{z}^j_{t|t})\,(\boldsymbol{z}_t - \boldsymbol{z}^j_{t|t})' \mid r_{1:t}, S_t = j, S_t = j], \tag{12}$$

and the covariance matrices:

$$\boldsymbol{\Omega}^{i(j)}_{t|t} = \mathbb{E}[(\boldsymbol{z}_t - \boldsymbol{z}^j_{t|t})\,(\boldsymbol{z}_t - \boldsymbol{z}^j_{t|t})' \mid r_{1:t}, S_t = j, S_{t-1} = i], \tag{13}$$

$$\boldsymbol{\Omega}^{(j)k}_{t|\tau} = \mathbb{E}[(\boldsymbol{z}_t - \boldsymbol{z}^{j(k)}_{t|\tau})\,(\boldsymbol{z}_t - \boldsymbol{z}^{j(k)}_{t|\tau})' \mid r_{1:\tau}, S_t = j, S_{t+1} = k], \tag{14}$$

where: $\boldsymbol{z}^{j(k)}_{t|\tau} = \mathbb{E}[\boldsymbol{z}_t \mid r_{1:\tau}, S_t = \mathrm{j}, S_{t+1} = k]$. From the prediction step of the switching Kalman filter (Saidane and Lavergne, 2007) we obtain the time updating formula for the common latent factors, $\boldsymbol{z}^{i(j)}_{t|t-1} = \boldsymbol{0}$, $\forall\ i, j = 1, \dots, n$ and their corresponding covariance matrices, $\boldsymbol{\Omega}^{i(j)}_{t|t-1} = diag[\omega^{i(j)}_{l,t|t-1}]$, where $\omega^{i(j)}_{l,t|t-1} = \beta^j_{0l} + \beta^j_{1l}\, z^i_{l,t-1|t-1} + \beta^j_{2l}[z^{i2}_{l,t-1|t-1} + \omega^i_{l,t-1|t-1}] + \beta^j_{3l}\, \omega^i_{l,t-1|t-2}$, for $l = 1, \dots, q$. Given the information set $D_{1:t-1} = \{r_{1:t-1}, Z_{1:t-1}, S_{1:t-1}\}$, the predicted variances are calculated as the conditional expectations of the predicted volatilities, $\mathbb{E}(\omega_{l,t} \mid D_{1:t-1})$, and from the total variance formula $\mathbb{E}(z^{i2}_{l,t-1} \mid D_{1:t-1}) = Var(z^i_{l,t-1} \mid D_{1:t-1}) + \mathbb{E}(z^i_{l,t-1} \mid D_{1:t-1})^2 = \omega^i_{l,t-1|t-1} + z^{i2}_{l,t-1|t-1}$, we obtain the filtered variances $\omega^i_{l,t-1|t-1}$. When a novel observation r_t becomes available, all the prediction estimates can be updated recursively via the Kalman filtering equations:

$$z^{i(j)}_{t|t} = z^{i(j)}_{t|t-1} + K_t(i,j)[r_t - \mu_j - \Phi_j\, z^{i(j)}_{t|t-1}], \tag{15}$$

$$\Omega^{i(j)}_{t|t} = [I_k - K_t(i,j)\Phi_j]\, \Omega^{i(j)}_{t|t-1} = \Omega^{i(j)}_{t|t-1} - K_t(i,j)\Gamma^{i(j)}_{t|t-1} K_t(i,j)', \tag{16}$$

with $\Gamma^{i(j)}_{t|t-1} = \Lambda_j + \Phi_j\, \Omega^{i(j)}_{t|t-1}\, \Phi_j'$ and $K_t(i,j) = \Omega^{i(j)}_{t|t-1}\, \Phi_j'\, \Gamma^{i(j)-1}_{t|t-1}$. The innovation cost $\delta_{t,t-1,i,j}$ related to each transition from state i to state j, is given by:

$$\delta_{t,t-1,i,j} = \frac{1}{2}\log|\Gamma^{i(j)}_{t|t-1}| + \frac{1}{2}[r_t - \mu_j - \Phi_j\, z^{i(j)}_{t|t-1}]'\, \Gamma^{i(j)-1}_{t|t-1}[r_t - \mu_j - \Phi_j\, z^{i(j)}_{t|t-1}] - \log p_{ij}. \tag{17}$$

A substantial part of this cost is exclusively due to the transition of the latent factors, as illustrated by the innovation component in Formula (17). The remaining part ($-\log p_{ij}$) reflects the transition of the market state from regime i to regime j. In this case, the minimization of the global cost at time t requires the selection of the optimal previous market state i: $\delta_{t,j} = \min_i\{\delta_{t,t-1,i,j} + \delta_{t-1,i}\}$. The resulting index is then recorded in the regime switching record, $\lambda_{t-1,j} = \arg\min_i\{\delta_{t,t-1,i,j} + \delta_{t-1,i})\}$. As a result, we obtain for each time t the "optimal" filtered latent factors $z^j_{t|t} = z^{\lambda t-1,j(j)}_{t|t}$ and their corresponding variances $\Omega^j_{t|t} = \Omega^{\lambda t-1,j(j)}_{t|t} = diag[\omega^{\lambda t-1,j(j)}_{l,t|t-1}]$.

When all the log-returns $r_{1:T}$ become available, we obtain the optimal global cost $\delta_T^* = \min_j\{\delta_{T,j}\}$. Then, we use the index of the optimal final state in order to decode the optimal sequence of HMM states: $j_T^* = \arg\min_j\{\delta_{T,j})\}$. To get the best regime for all time steps, we trace back through the market regime switching record: $j_T^* = \lambda_{t,j^*_{t+1}}$.

We note here that the smoothing gain matrix $L_t^{(j)k} = \Omega^j_{t|t}\, 0_k'\Omega^{(j)k-1}_{t+1|t} = 0$ and the smoothing equations are simply given by:

$$z^{(j)k}_{t|T} = z^j_{t|t} + L_t^{(j)k}[z^k_{t+1|T} - z^{j(k)}_{t+1|t}] = z^j_{t|t}, \tag{18}$$

$$\Omega^{(j)k}_{t|T} = \Omega^j_{t|t} + L_t^{(j)k}[\Omega^k_{t+1|T} - \Omega^{j(k)}_{t+1|t}]\, L_t^{(j)k'} = \Omega^j_{t|t}. \tag{19}$$

Following the smoothing procedure developed by Saidane and Lavergne (2008), the sufficient statistics for our estimation problem will be given by: $\mathbb{E}(S_t|\cdot) = S_t(j^*)$, $\mathbb{E}(S_t\, S_{t-1}'|\cdot) = S_t(j^*)\, S_{t-1}(j^*)'$ and $\mathbb{E}(z_t\, S_t(j)|\cdot) = z^{jt^*}_{t|T}$), if $j = j_t^*$ and $\mathbf{0}$ otherwise. In this case, the operator $\mathbb{E}(|\cdot)$ denotes the expectation with respect to the distribution $p(Z, S|r)$.

3 MAXIMUM LIKELIHOOD ESTIMATION

We propose a two-step learning algorithm combining the expectation maximization (EM) algorithm (Dempster et al., 1977) and the Viterbi decoding algorithm in order to estimate the parameters Θ of our model. The E-step subsists in calculating the expected value of the complete data log-likelihood function with respect to the conditional distribution of the unobserved variables (Z, S) given the observed returns r and $\Theta^{(e)}$, the value of the parameter at the current iteration (e). The conditional expectation is then, maximized with respect to Θ at the M-step. In this case, the auxiliary function that will be maximized can be approximated as follows:

$$\begin{aligned} Q(\Theta,\Theta^{(e)} \simeq{}& \sum_{j=1}^{n} S_1(j)\log p(S_1) - \sum_{t=2}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n} S_t(j)\, S_{t-1}(i)\log p_{ij} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} S_t(j)[\log|\Lambda_j| + \mathbb{E}\{(r_t - \mu_j - \Phi_j\, z^j_t)' \\ & \Lambda_j^{-1}(r_t - \mu_j - \Phi_j\, z^j_t)|r_{1:T}), \Theta^{(e)}\}] - \frac{1}{2}\sum_{j=1}^{n}\sum_{l=2}^{q}\sum_{t=1}^{T} S_t(j)\, \mathbb{E}[\log(\omega^j_{l,t}) + \frac{z^2_{l,t}}{\omega^j_{l,t}}|r_{1:T}, \Theta^{(e)}], \end{aligned} \tag{20}$$

and the conditional expectations can be derived using the sufficient statistics obtained by the Viterbi algorithm in Section 3.

The basic idea behind our algorithm is summarized as follows: At the end of each iteration (e) we find $\Theta^{(e+1)}$, the optimal value of the parameter Θ that maximizes the function in equation (20) over all possible values of Θ. Then $\Theta^{(e+1)}$ replaces $\Theta^{(e)}$ in the E-step and $\Theta^{(e+2)}$ is chosen to maximize $Q(\Theta,\Theta^{(e+1)})$, and so on until convergence. However, given the nonlinear dependency of the common variance parameters in the last summation of Formula (20), we can maximize in a first time this function with respect to the probabilities of the initial state π_j, the transition probabilities p_{ij}, the specific means $\boldsymbol{\mu}_j$, the factor loadings $\boldsymbol{\Phi}_j$ and the specific variances $\boldsymbol{\Lambda}_j$. In a second time, the parameters of the common variances can be determined numerically.

For the initial state probabilities π_j, we use the Lagrange multipliers approach subject to the condition that the sum $\sum_{j=1}^{n} \pi_j = 1$, and we obtain the updated estimation:

$$\hat{\pi}_j = \frac{S_1(j)}{\sum_{j=1}^{n} S_1(i)} . \tag{21}$$

We use also the Lagrange formalism, subject to the unity constraint $\sum_{j=1}^{n} p_{ij} = 1$, to obtain the updated transition probabilities:

$$\hat{p}_{ij} = \frac{\sum_{t=2}^{T} S_t(j) S_{t-1}(i)}{\sum_{t=2}^{T} S_{t-1}(i)} . \tag{22}$$

The maximization of the auxiliary function with respect to the specific means yields the updated estimates:

$$\hat{\boldsymbol{\mu}}_j = \frac{1}{\sum_{t=1}^{T} S(j)} \sum_{t=1}^{T} S_t(j)(\boldsymbol{r}_t - \boldsymbol{\Phi}_j \, \boldsymbol{z}_{t|T}^{j}) . \tag{23}$$

The updated l-th row of the factor loadings matrix $\hat{\boldsymbol{\Phi}}_j$ can be expressed as follows:

$$\hat{\boldsymbol{\phi}}_{l,j} = \left[\sum_{t=1}^{T} S_t(j)(r_{l,t} - \mu_{l,j}) \boldsymbol{z}_{t|T}^{j}\right]' \left[\sum_{t=1}^{T} S_t(j)[\boldsymbol{\Omega}_{t|T}^{j} + \boldsymbol{z}_{t|T}^{j} \boldsymbol{z}_{t|T}^{j'}]\right]^{-1}, \tag{24}$$

where: $\mu_{l,j}$ is the specific mean of the l-th asset return $r_{l,t}$ under the market regime j. Then, given these updated parameters, we can update the specific variances according to the following rule:

$$\hat{\boldsymbol{\Lambda}}_j = \frac{1}{\sum_{t=1}^{T} S(j)} \sum_{t=1}^{T} S_t(j) diag\,[\boldsymbol{\Phi}_j \boldsymbol{\Omega}_{t|T}^{j} \boldsymbol{\Phi}_j' + (\boldsymbol{r}_t - \boldsymbol{\mu}_j - \boldsymbol{\Phi}_j \boldsymbol{z}_{t|T}^{j})(\boldsymbol{r}_t - \boldsymbol{\mu}_j - \boldsymbol{\Phi}_j \boldsymbol{z}_{t|T}^{j})]' . \tag{25}$$

In a second time, given the new values of π_j, p_{ij}, $\boldsymbol{\mu}_j$, $\boldsymbol{\Phi}_j$ and $\boldsymbol{\Lambda}_j$, we can approximate the conditional distribution of the log-returns by the normal distribution: $\boldsymbol{r}_t \,|r_{1:t-1}, S_t = j, S_{1:t-1} \sim N[\boldsymbol{\mu}_j, \boldsymbol{\Gamma}_{t|t-1}^{j}]$ (e.g. Harvey et al., 1992). In this case, $\boldsymbol{\Gamma}_{t|t-1}^{j} = \boldsymbol{\Lambda}_j + \boldsymbol{\Phi}_j \, \boldsymbol{\Omega}_{t|t-1}^{j} \, \boldsymbol{\Phi}_j'$ and $\boldsymbol{\Omega}_{t|t-1}^{j} = diag[\omega_{t|t-1}^{\lambda_{t-1,j}(j)}]$ is the conditional expectation of $\boldsymbol{\Omega}_t$, given the sequences $r_{1:t-1}$ and $S_{1:t-1}$, obtained via the modified Kalman filter approach based on the Viterbi decoder developed in Section 3.

Using these approximations and ignoring the initial conditions, we obtain the following pseudo log-likelihood function:

$$L^* = c - \frac{1}{2}\sum_{t=1}^{T}\sum_{j=1}^{n} S_t(j)[\log|\boldsymbol{\Gamma}^{j-1}_{t|t-1}| + (\boldsymbol{r}_t - \boldsymbol{\mu}_j)'\boldsymbol{\Gamma}^{j-1}_{t|t-1}(\boldsymbol{r}_t - \boldsymbol{\mu}_j)] \ . \tag{26}$$

In a first stage, we ignore the elements in the last summation of Formula (20) and then we maximize the remaining terms with respect to (π_j, p_{ij}, $\boldsymbol{\mu}_j$, $\boldsymbol{\Phi}_j$ and $\boldsymbol{\Lambda}_j$) using the EM algorithm. During this step, the parameters of the quadratic GARCH processes $\beta = \{\beta_0, \beta_1, \beta_2, \beta_3\}$ are kept fixed to their values obtained in the previous iteration. In a second stage, we optimize the pseudo log-likelihood in Formula (26) with respect to β, using the values of π_j, p_{ij}, $\boldsymbol{\mu}_j$, $\boldsymbol{\Phi}_j$ and $\boldsymbol{\Lambda}_j$ found in the first step. The R package NlcOptim, developed by Chen and Yin (2019), can be used in this step to find quickly and most accurately the parameters of the conditionally heteroskedastic component β.

4 THE FHMV APPROACH FOR VALUE-AT-RISK PREDCTION

Formally put, Value-at-Risk is a financial metric that measures the worst expected loss that could happen in an investment portfolio over a given horizon for a given confidence level. In this section a general Monte Carlo simulation FHMV-based framework for value-at-risk prediction, under regime switching dynamics, will be proposed. This approach will then be used, in Section 5, for the evaluation of the currency risk associated with the Tunisian government's foreign debt portfolio during the revolution period of 14 January 2011.

4.1 Forecasting future market regime changes

Given the information set available at time t, $D_{1:t}$, and the actual market regime i, the conditional mean of the multivariate predictive distribution given by our FHMV model is as follows:

$$\mathbb{E}(\boldsymbol{r}_{t+1}|D_{1:t}) = \boldsymbol{\mu}_j , \tag{27}$$

and the conditional variance-covariance matrix is given by:

$$\boldsymbol{\Gamma}^j_{t+1|t} = \boldsymbol{\Lambda}_j + \boldsymbol{\Phi}_j\boldsymbol{\Omega}^j_{t+1|t}\boldsymbol{\Phi}_j' \ . \tag{28}$$

Within this framework, the forecasts of the future market regime jumps and the model parameters updating process are implemented simultaneously. Thus, by the end of each transaction day the closing prices will be included in the database. Thereafter, the parameters of our model will be updated using the newer information set available at this point in time, and the updated one-step-ahead forecasts of the common latent factor variances will be derived via the relation: $\tilde{\omega}^j_{l,t+1|t} = \beta^j_{0l} + \beta^j_{1l}z^i_{l,t|t} + \beta^j_2 z^{i2}_{l,t|t} + \beta^j_3\omega^i_{l,t|t}$. Then, the future market regime $S_{t+1} = j$ can be obtained as a solution of the optimization problem: $\hat{S}_{t+1|t} = \arg\max_j p(S_{t+1} = j \mid S_t = i_t^*)$, where i_t^* is the optimal market regime at time t obtained by the Viterbi algorithm, through the state transition record $\lambda_{t-1,i}$ at each time step. The FHMV model with the optimal future hidden state $\hat{S}_{t+1|t}$, will be used in the simulation procedure as the data generating process, to calculate the VaR of our portfolio.

4.2 The simulation strategy

Our simulation strategy consists of the following steps:

1. Firstly, we define the coverage rate α of the VaR.
2. Then, taking into account the presence of leverage effects and conditional skewness in financial time series, we simulate different return scenarios from the conditional distribution of the common latent factors $\boldsymbol{z}^s_{t+1|t}$, using the optimal specification obtained by the Viterbi algorithm at time t (Section 5.1).
   a. We use in a first time the normal distribution $N(\boldsymbol{0},\boldsymbol{I}_q)$ to generate the standardized factors $\boldsymbol{z}^*_{t+1}$.

b. Then, we compute the lower triangular Cholesky factor $\mathbf{\Omega}^{j*}_{t+1|t}$ of the variance-covariance matrix $\mathbf{\Omega}^{j}_{t+1|t}$, and we obtain: $\boldsymbol{z}^{s}_{t+1|t} = \mathbf{\Omega}^{j*}_{t+1|t}\, \boldsymbol{z}^{*}_{t+1}$.

3. After that, we simulate different return scenarios from the conditional distribution of the specific factors $\epsilon^{s}_{t+1|t}$, using also the optimal specification obtained by the Viterbi algorithm at time t.
   a. We generate in a first time from the normal distribution $N(\mathbf{0},\boldsymbol{I}_p)$ the standardized specificities $\epsilon^{*}_{t+1|t}$.
   b. Then, we compute the lower triangular Cholesky factor $\mathbf{\Lambda}_j^{*}$ of the variance-covariance matrix $\mathbf{\Lambda}_j$, and we obtain: $\epsilon^{s}_{t+1|t} = \mathbf{\Lambda}_j^{*}\, \epsilon^{*}_{t+1}$.
4. In the fourth step, we compute m different portfolio's returns for the period $t + 1$ as, $R_{s,t+1|t} = \gamma_1\, r^{s}_{1,t+1|t} + \gamma_2\, r^{s}_{2,t+1|t} + \cdots + \gamma_p\, r^{s}_{p,t+1|t}$, where $\gamma_1, \gamma_2, \ldots, \gamma_p$ denote the portfolio weights of the p risk factors and $\boldsymbol{r}^{s}_{t+1|t} = \boldsymbol{\mu}_j + \mathbf{\Phi}_j\, \boldsymbol{z}^{s}_{t+1|t} + \epsilon^{s}_{t+1|t}$ $(\forall s = 1, \ldots, m)$.
5. Finally, to compute the portfolio's VaR for the period $t + 1$, we sort the simulated values in ascending order and we exclude the α% lowest returns $R_{s,t+1|t}$. In this case, the predicted VaR is the minimum of the remaining returns.

5 NUMERICAL EXPERIMENTS USING EXCHANGE RATE DATA

In this empirical experiment, we use the FHMV model to analyze the dynamic latent correlation structure of the five dominant currencies in the Tunisian government's foreign debt portfolio. The optimal specification obtained with the Viterbi algorithm will then be used to evaluate, through a backtesting exercise, the performance of the new methodology in detecting the foreign exchange risk associated with this portfolio. All the numerical results and the graphs in this section are obtained using the R statistical freeware, version 4.1.

5.1 Data presentation and summary

We focus in this section on the main five currencies forming the basis for the Tunisian government's foreign debt portfolio, namely the European euro (EUR), the American dollar (USD), the Japanese yen (JPY), the Swiss franc (CHF) and the British pound (GBP). Our dataset, downloaded from the Yahoo

Table 1 Basic descriptive characteristics of the daily exchange rate returns from 2/1/2010 to 30/12/2012

Statistic	EUR/TND	USD/TND	JPY/TND	CHF/TND	GBP/TND
Mean	0.000327	0.000381	0.000179	0.000274	0.000363
Max	0.0439	0.0423	0.0514	0.0424	0.1468
Median	0.000198	0.000255	0.000163	0.000308	0.000212
Min	–0.0538	–0.0619	–0.0718	–0.0653	–0.0083
Std. Dev.	0.00425	0.00510	0.00674	0.00583	0.00620
D'Agost. test	7.49623	2.72961	4.14286	–6.21753	9.57321
	(0.0000)	(0.0047)	(0.0000)	(0.0081)	(0.0000)
A-Glyn. test	18.751	16.173	15.927	16.369	21.916
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
LB. test	117.67	108.21	41.962	62.114	123.813
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
J-Bera. test	65.321	29.655	14.533	26.123	34.259
	(0.0000)	(0.0000)	(0.0074)	(0.0000)	(0.0000)

Note: The values into brackets represent the p-values of the corresponding tests.
Source: Own construction

Finance website, spread over the period between 2/1/2007 and 30/12/2012, consists of 1 500 daily exchange rates for the different currencies expressed in terms of Tunisian dinar (TND). This dataset includes the period of social mobilization and political change in Tunisia (the revolution of 14 January 2011). In this case, taking into account the period of social instability we will be permitted to investigate the efficiency of our Jump-VaR methodology during crisis times.

In Table 1, we give a variety of descriptive statistics to study the distributional characteristics of the data and to test the empirical skewness and Kurtosis against the values of normal distributions (e.g. D'Agostino, 1970; Anscombe-Glynn, 1983). We implemented also the normality test (Jarque-Bera, 1980). From these results, we note that all the log-returns are non-normally distributed, they are still skewed (positive for EUR, USD, JPY and GBP and negative for CHF). We note also a positive excess kurtosis for all the currencies. The results of the Ljung-Box (1978) statistic show the presence of volatility clustering. This imply that we have a non-constant conditional volatility, and the use of a Markov-switching specification with a time-varying co-movement structure for the log-return series, is more realistic in this situation.

5.2 A preliminary latent structure analysis of the data

In order to select the most appropriate model that fits better our dataset, we used the Akaike (AIC) and the Bayesian (BIC) information criteria. To this end, we trained standard and conditionally heteroskedastic models using one or two common factors and a number of hidden states varying between one and three, on the period from 2/1/2010 to 30/12/2012. Then, we used the selection criteria to identify the best model with the minimum AIC and BIC values.

Table 2 AIC and BIC values for the different specifications over the period 2/1/2010–30/12/2012 (750 observations)

The number of common factors	Criterion	Number of hidden states		
		1	2	3
1	AIC	2 436.2	1 973.6	2 199.2
		(2 826.9)	(2 610.5)	(2 585.9)
	BIC	2 518.3	2 424.4	2 591.0
		(3 097.6)	(2 731.8)	(2 644.3)
2		2 394.8	1 958.3	2 123.8
		(3 016.4)	(2 415.2)	(2 275.5)
		2 509.6	2 151.8	2 269.9
		(3 211.1)	(2 597.8)	(2 403.7)

Note: The selection criteria values for the standard models are given into brackets.
Source: Own construction

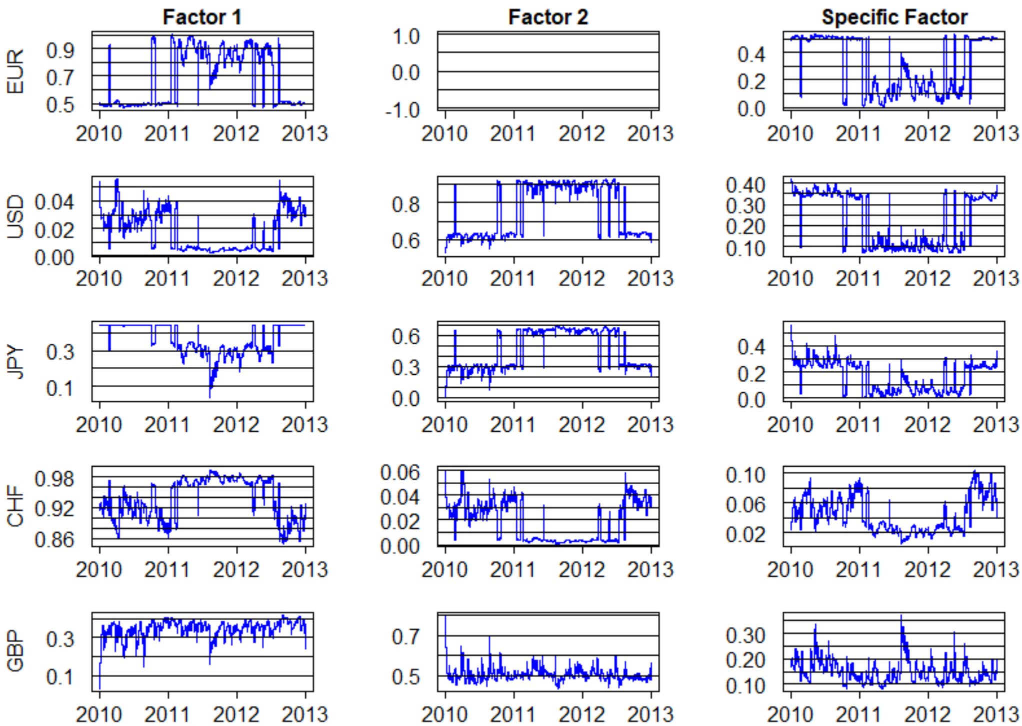
The results reported in Table 2 show that the FHMV model with two common factors and two HMM states is the best one fitting our dataset. For this optimal specification, the initial state probability vector and the transition probability matrix are as follows:

$$\pi = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{P} = \begin{bmatrix} 0.9491 & 0.0509 \\ 0.2311 & 0.7689 \end{bmatrix}.$$

In Figure 1, we depict the percentage of the variability of the different currencies expressed in terms of specific and common factors. During the crisis period, we can see that, on average, 90% and 95% of the variances of the EUR and the CHF are explained by the first common factor. During the normal period (before and after the 2011–2012), we can see that the first common factor explains on average 50% and 90% of the variability of the EUR and CHF.

The contribution of the second common factor to the variability of the EUR and CHF, during the instability period, is almost insignificant. During the revolution period, this factor explains more than 85% and 65% of the variability of the USD and JPY. However, the contribution of the first factor to the JPY variability is around 45% during the normal period. For the GBP, the contribution of this factor is around 35% over the whole period.

Figure 1 The percentage of variability of the different log-returns associated to the common and specific factors over the period 2/1/2010–30/12/2012



Source: Own construction

From these results, we can conclude that the first common factor is associated with the volatility dynamics of the European currencies. During the social mobilization period, the second factor is associated with the volatility dynamics of the American and Japanese currencies. We can conclude also that the first common latent factor expresses the relative value of the TND against the major trading partner's currencies (the European community countries). The second factor reproduces the relative value of the TND against a basket of global currencies in which the American and Japanese currencies are dominant.

From the estimation results presented in Table 3, we note that the first common factor can be regarded as a European factor: it represents a basket of currencies, where the EUR dominates with relatively high loadings (50% in the first regime and 76% in the second regime). The weight of the GBP is relatively reduced in this basket. We note also that the second common factor represents a basket of currencies, where the USD dominates with relatively high loadings (52% in the first regime and nearly 60% in the second regime). In order to satisfy the identification constraints (e.g. Saidane and Lavergne 2011), we have taken $\phi_{1,2,j} = 0$, $\forall_j = 1, 2$, which imply that the European currency EUR is entirely absent from the second factor. The relative weight of the CHF is also reduced in this factor. Hence, we can consider the second common factor as an American factor.

Table 3 Estimation results of the optimal FHMV model during the period 2/1/2010–30/12/2012

Model parameters (1e-04)	Currencies				
	EUR	USD	JPY	CHF	GBP
$\hat{\mu}_1$	1.6104	1.5216	1.1622	–1.1131	1.3109
$\hat{\mu}_2$	2.1724	3.0221	1.7447	0.0156	1.7932
$\hat{\Phi}_1$	7.7028	1.9780	1.2636	2.6804	1.7611
	0.0000	5.1303	2.3204	0.3217	2.1032
$\hat{\Phi}_2$	9.1526	0.7315	0.3949	1.4710	0.2389
	0.0000	8.9482	2.5801	0.6206	2.8312
$\hat{\Lambda}_1$	0.6405	1.0623	1.5049	1.1458	2.1004
$\hat{\Lambda}_2$	0.2230	0.9071	1.2047	1.0167	2.7103

Source: Own construction

Table 4 Estimated parameters of the conditionally heteroskedastic components of the optimal FHMV model during the period 2/1/2010–30/12/2012

Common Variance parameters	β_0	β_1	β_2	β_3
	Factor 1			
Regime 1	0.1224	0.0009	0.2411	0.3648
Regime 2	0.1521	0.1382	0.2363	0.7601
	Factor 2			
Regime 1	0.0843	0.1774	0.1892	0.4667
Regime 2	0.1102	0.1430	0.2711	0.7092

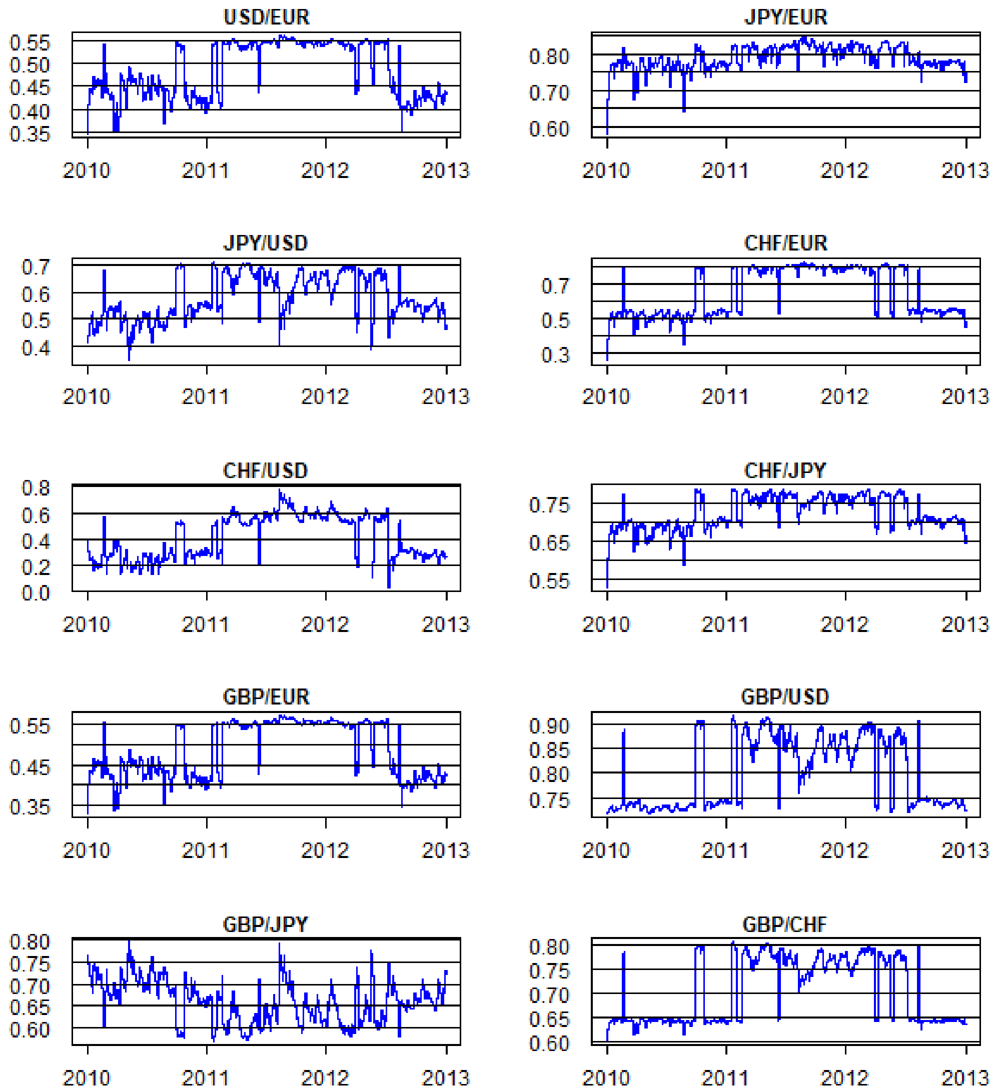
Source: Own construction

From Table 4, it appears that the excess of volatility during the political instability period (the second hidden regime) is relatively due to the significant increase in volatility persistence (e.g. Klaassen, 2002). We observe from this table that the sum of the volatility parameters, β_2 and β_3, of the two common factors in the second regime is nearly close to 1.

All the previous conclusions are strongly confirmed by the estimated values of the specific variances, given in Table 3. Hence, during the social mobilization period, the specific variance of the British pound is relatively high, which indicates its aberration from its latent factorial class. On the other hand, the specific variances of the European euro and the American dollar are the smallest ones, which indicate their determinant role in their latent factorial class.

Finally, in Figure 2, we depict the correlation structure of the different log-returns during the period 2/1/2010 to 30/12/2012. The graph picks up co-movement increases between all the log-returns from the beginning of 2011 until near the end of the study period. This result confirms the financial contagion that affected the Tunisian economy during the revolution period.

Figure 2 Plot of the estimated correlations from the best FHMV model, during the period 2/1/2010–30/12/2012



Source: Own construction

5.3 Selection of the most appropriate VaR model

In order to assess the currency risk associated with the Tunisian government's foreign debt portfolio during the social mobilization period of 14 January 2011, we divided in a first time our dataset into calibration set and test set. The calibration, called also training, set contains the log-returns of the different exchange rates during the period 2/1/2007–30/12/2009 (750 observations). The test, called also backtesting, set contains the remaining 750 observations covering the period 2/1/2010–30/12/2012. Then, we used the Monte Carlo simulation strategy (Section 4.2) to evaluate the VaR of our portfolio. For each coverage rate α, we used the portfolio weights given in Table 5. Here, the weight of each exchange rate γ_k is determined by the relative share of currency k in the payment of the total foreign debt. For example, in 2010 Tunisia settled 61.3% of its foreign loans in Euro, 14.3% in American dollar, 16.1% in Japanese yen, 2.4% in Swiss franc and 5.9% in British pounds. Hence, in 2010, $\gamma_1 = 0.613$, $\gamma_2 = 0.143$, $\gamma_3 = 0.161$, $\gamma_4 = 0.024$ and $\gamma_5 = 0.059$. For 2011 and 2012, the weights are determined in the same way.

Table 5 Structure of the Tunisian foreign debt by settlement currency for the period 2010–2012

Date	Indicators				
	EUR	USD	JPY	GBP	CHF
31/12/2010	61.3	14.3	16.1	5.9	2.4
31/12/2011	56.8	20.1	15.3	5.6	2.2
31/12/2012	59.6	18.9	13.8	5.1	2.6

Source: Monetary and financial statistics of the Tunisian central bank

In a second time, the effectiveness of our methodology is justified by some experiments, using unconditional (Kupiec, 1995) and conditional (Christoffersen, 2012) tests and the rolling sample method based on a one-day moving window scheme with the coverage rates from the level of 0.005 to 0.1 by 0.005. All these calculations have been carried out by simulations from our FHMV model, the mixed factorial hidden Markov model (MFHMM) by Saidane (2019), the latent factor model with time varying volatility (FM) by Saidane (2017) and the classical Monte Carlo simulation method (CMC) by Mosbahi et al. (2017).

In order to compromise between precision and efficiency, we generated m = 25 000 scenarios from each competing model (e.g. Saidane, 2022; Lu et al., 2014; Bastianin, 2009; Fantazzini, 2008). Then, we calculated the VaR, the exception rates and the likelihood ratios for the proportion of failure test (LR-*pof*), the independence test (LR-*ind*) and the conditional coverage test (LR-*cc*).

All the results of the backtesting experiments are given in Table 6 and Figures 3–4. The Kupiec and Christoffersen backtesting results show that the optimal FHMV model, with 2 latent factors and 2 hidden states, provides good results and gives exception rates very close to the target (the true coverage rates α). The likelihood ratios associated with the unconditional and independence tests, for our proposed model, are always lower than the critical values, which imply a significant conditional coverage tests for all the confidence levels.[2]

Table 6 Backtesting results for the FHMV model with two hidden states and two common factors

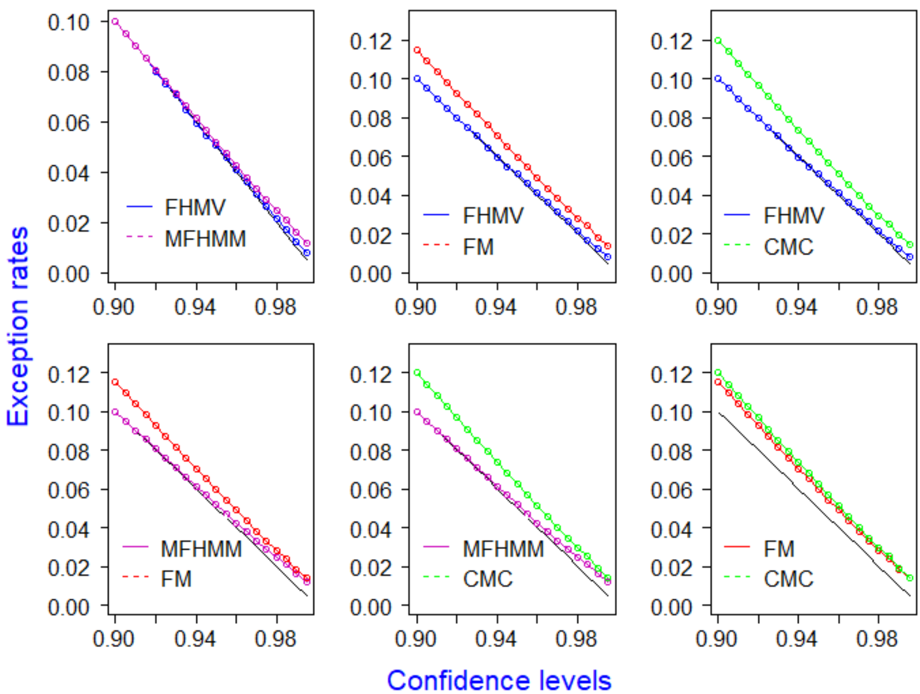
Confidence level	Exception rate	1st exception	LR-pof	LR-ind	LR-cc
0.950	0.050	18	1.7934	1.1152	2.9086
0.960	0.040	22	1.8372	1.1256	2.9628
0.970	0.029	38	2.0839	1.1398	3.2237
0.980	0.022	38	2.1856	1.1458	3.3314
0.990	0.013	56	2.3122	1.1593	3.4715

Source: Own construction

[2] The critical values for the Kupiec and Christoffersen tests are, respectively, χ^2 (1) = 3.8414 and $\chi^2(2)$ = 5.9915 for 95% VaR.

For the coverage rates from 0.5% to 2%, Figure 3 shows promising results for the optimal FHMV model compared to those given by the best MFHMM (with 2 mixture components and 2 latent factors). For the significance level 2%, our FHMV model gives, for example, an exception rate equal to 2.17%, versus 2.45% obtained by the MFHMM. From this figure, we can see also that the optimal MFHMM looks better than the FM and CMC, especially at low confidence levels. Hence, we can argue that the FHMV model is the more precise and yields higher-quality predictions, as compared to the other competing models.

Figure 3 Exception rates for various confidence levels from the rolling window experiments

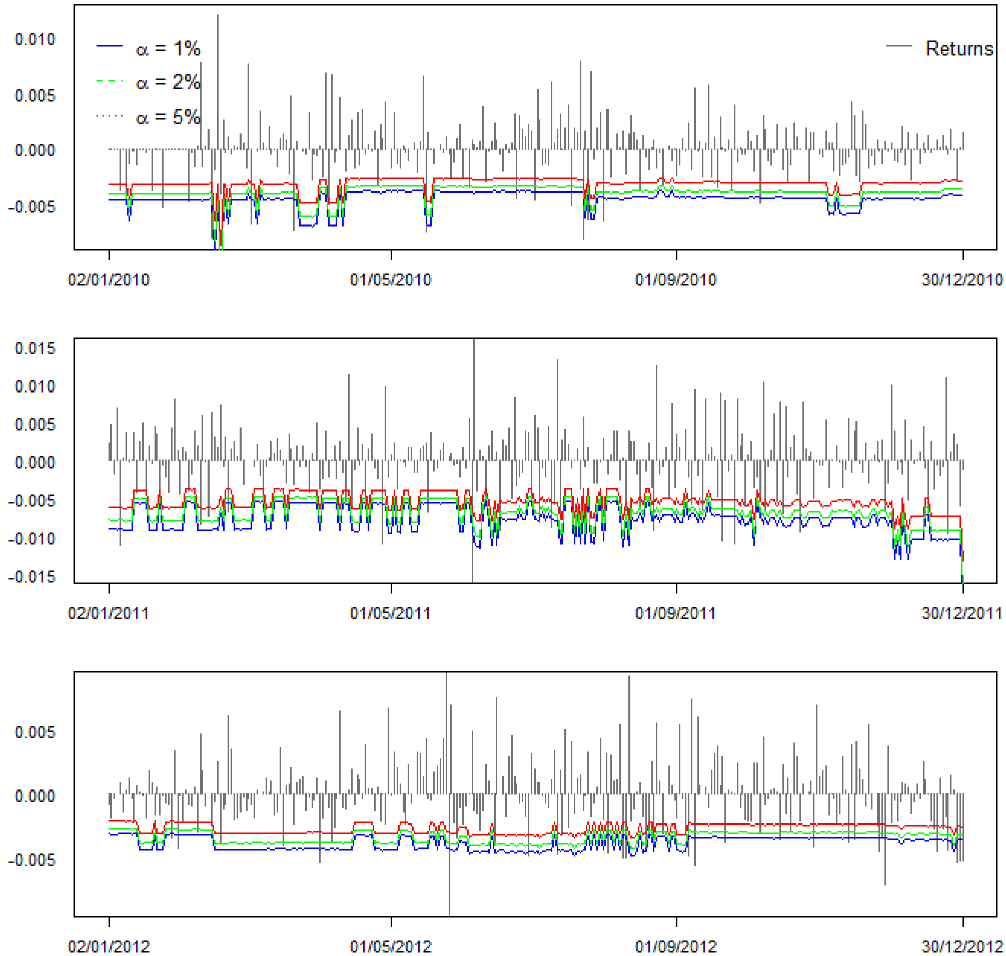


Source: Own construction

In order to compare the results given by the different models, we used the squared relative prediction error criteria $S = \sum_{i=1}^{20}[(E_i - \alpha_i)/\alpha_i]^2$, where E_i are the estimated exception rates obtained with the different specifications, and α_i the coverage rates. It appears from the results that the most adequate model to evaluate the VaR of our portfolio, during this period, is the FHMV framework. This specification gives the estimated exception rates closest to all the true significance levels with $S = 0.4561$. The second ranked model is the MFHMM ($S = 2.5824$), the third is the FM with ($S = 4.9783$), and the CMC is the worst one with $S = 6.0673$.

Finally, we can see from Figure 4 the significant effect of the volatility shocks on the predicted VaR given by the optimal FHMV model. Hence, we can argue that the major reason for the bad results given by the CMC, FM, and to a lesser degree the MFHMM, is that they do not take into account the abnormal switching behaviors, which can affect the volatility and the co-movement dynamics in financial markets during crisis periods.

Figure 4 The VaR exceptions obtained by the optimal FHMV model for different coverage rates and different portfolio weights



Source: Own construction

CONCLUSION

This paper develops a new multivariate approach for Value-at-Risk (VaR) prediction. Our strategy considers the possibility of regime jumps in the intra-portfolio's latent correlation structure and allows for time-varying volatility in the factor variances. The proposed framework combines factor analysis models with GQARCH processes and hidden Markov models. During financial crisis periods, this specification provides a more tractable way to capture simultaneously the switching interrelations between assets and the time-varying volatility of each individual asset.

The accuracy of the new prediction approach in comparison with other existing models (such as the mixed factorial hidden Markov model, the latent factor model with time varying volatility and the classical Monte Carlo method) is evaluated through a real dataset example from the Tunisian foreign exchange market for the period 2/1/2010 to 30/12/2012. Our strategy aims to select the best model that could predict the VaR of the Tunisian government's foreign debt portfolio during the social mobilization period of 14 January 2011. In that period the Tunisian economy has experienced the longest, deepest and most broad-based recession in its history since the 1978. The main results of the empirical example and the backtesting experiments, based on the rolling sample method, show that the new approach appears to give a good fit to the data, allows to more close forecasts to the market changes and can improve the VaR predictions and offer more accurate VaR estimates than the other competing models for all coverage rates from 0.5% to 10%.

We conclude that our Viterbi-based decoding strategy using the factorial hidden Markov volatility model seems to be a useful tool for portfolio risk management and control, especially during periods of financial market stress. These results support our argument for integrating time-varying volatility and regime jumps into the risk measurement framework. In the forthcoming works, we intend to reflect the interaction between the common latent factors with a dynamic structure for the idiosyncratic variances. We will address also nonlinear behaviors, non-homogeneous transition probabilities and other areas of application, like options, or credit derivatives.

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